

One-sided classical-register dynamics: positive divisibility forbids entanglement, complete positivity at all times does not

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Abstract

No-go theorems state that two quantum systems which interact only through a classical mediator cannot become entangled. In their LOCC-type forms the mediation hypothesis is imposed on channels, or on rounds of channels. We ask what that hypothesis requires of a continuous-time dynamics whose generator is one-sided at every instant and whose mediator is a finite classical register. For bounded generators we prove (paper proofs, not machine-checked) that divisibility suffices. If every intermediate map is completely positive (CP-divisibility), register-conditional separability is preserved. For inputs without ancillas it is enough that every intermediate map is positive (P-divisibility), and even that it is positive on product states tested with product measurements; this sharper statement was checked at the level of the written proof by an independent model of another provider. Its proof does not cover inputs with local ancillas, for which we rely on CP-divisibility. Divisibility cannot be weakened to complete positivity at all times. We give an explicit family in which two qubits are coupled through an eight-state classical register by one-sided generators, the evolution is completely positive and trace preserving at all times by a closed-form certificate that is formally verified in Lean 4 (with complete positivity read as positivity of the Choi matrix), and yet the qubits become entangled (Lean-verified PPT witness; negativity 0.388 from $|+\rangle|+\rangle$, numerically). This family is not P-divisible: an intermediate map sends a product state to an operator that is not positive. Its register is classical at each single time. If the same time-local law is assumed to govern the state after an intermediate reading, then reading the register, continuing the dynamics and measuring a qubit gives a negative joint quasi-probability. The mechanism is a which-path record written by one qubit and erased at the other by trace-preserving conditional maps that are not positive. Divisibility is sufficient but not necessary. We transplant the construction onto a lattice chain of classical cells shaped like Oppenheim’s classical-quantum master equation, keeping one-sidedness and site-local quantum jumps, with a declared nearest-neighbour classical transport kernel. This numerical supplement is not part of the proved results. As built, that family is not completely positive, with negative Choi eigenvalues of order ce^{-2R_2} in the flush rate R_2 . One added local depolarizing window gives a two-site member numerically consistent with complete positivity at the default finite rates, with negativity 0.386 unchanged; floating-point checks and an interval enclosure of one exposed-block margin do not certify the full family, so its complete positivity at all times remains conjectural. These are statements about formal hybrid dynamics, not about gravity. We make no claim that any physical theory of classical gravity contains these members. Tomizuka and Takeda motivate an analogous hidden-quantum interpretation; we do not construct a compatible quantum-environment dilation of these finite families here. The constructions and proofs were found and checked by AI agents; no human expert has yet verified them.

1 Introduction

Two proposals [1, 2] suggested that gravitationally induced entanglement between two masses, each prepared in a spatial superposition, would witness a nonclassical gravitational field. The inference rests on a no-go statement: two quantum systems that interact only with a classical mediator cannot become entangled. The statement has several formalizations. They include the inaccessible-mediator criteria of Krisnanda, Paterek and collaborators [3, 4, 5], a generalized-probabilistic-theory no-go [6], a theorem for mediators described by commutative C^* -algebras [7], and Di Biagio’s decomposition of the argument into a statespace assumption and a mediation assumption [8]. Counterpoints are also in print. Hall and Reginatto gave counterexamples in the configuration-ensemble formalism [9]. Trillo and Navascués showed that the Diósi–Penrose model predicts GIE [10]. Angeli and Carlesso found entangling Markovian hybrid models [11], and Ulbricht, Bermúdez Manjarres and Reginatto found one in hybrid van Hove theory [12]. Di Biagio now argues that GIE is not a theory-independent witness of nonclassicality [8]. A separate dispute, whether classical gravity in quantum field theory produces entanglement [13], concerns closed-system unitary dynamics and is not addressed here.

The assumption this paper isolates. In the LOCC-type statements [7, 8] mediation is a property of channels. The evolution is a sequence of rounds, or a limit of such sequences, and each round acts either on A and the mediator or on B and the mediator. Physical models, however, are specified by generators. Suppose a generator is one-sided at every instant, $L_t = L_t^A + L_t^B$. It yields round-mediated dynamics through a Trotter product if each piece generates completely positive (CP) maps, which is the case when the dynamics can be cut into small CP steps, that is, when it is CP-divisible [14, 15]. The theorem statements of Refs. [7, 8] are made at the channel level and do not name continuous-time divisibility. Their rounds of CP channels constrain the maps between round boundaries only; they do not imply that a continuous family is CP between arbitrary pairs of times. (Local dephasing of A with coherence multiplier $e^{-\sin t}$, $0 \leq t \leq \pi$, is a one-sided CPTP channel at every time, while its intermediate map from $t = \pi/2$ to $t = 3\pi/4$ multiplies coherences by $e^{1-\sin(3\pi/4)} \approx 1.34$ and is not even positive.) We therefore do not claim that CP-divisibility is a hidden hypothesis of those theorems, and no theorem we cite is contradicted by what follows. Other works state Markovianity or time locality explicitly as a hypothesis [11, 16]. The point that the step decomposition can fail is also in print. Trillo and Navascués write that “the dynamics of models of classical gravity do not necessarily decompose into sequences of pair-wise interactions between each massive particle and a classical field” [10], and in their model the obstruction is a two-body dissipator, so the generator is not one-sided. Tomizuka and Takeda note, for classical-quantum dynamics obtained from a decohered quantum mediator, that complete positivity “does not imply CP divisibility” [17]. The question we answer is narrower. If the generator is one-sided at every instant, the register is classical at each single time, and the evolution is CP at every time, does the absence of entanglement still follow without divisibility? And if some divisibility is needed, how much?

Results.

1. Theorem 1 (paper proof): for a finite classical register and locally bounded one-sided generators, CP-divisibility implies that register-conditional separability is preserved. Hence no entanglement is generated. Time-dependent families are allowed.
2. Theorem 2 (paper proof): for inputs without ancillas the same conclusion holds under P-divisibility, and under the weaker product-positive divisibility of Definition 5, with no trace-

preservation hypothesis. The hypothesis is strictly weaker than CP-divisibility on this class (Example 1). The one-sided factors that the proof builds are positive but need not be CP, so the proof does not cover inputs with local ancillas; for those, Theorem 1 is what we have.

3. Theorem 3: an explicit two-qubit, eight-state-register family satisfies every hypothesis of Theorem 1 except divisibility. It is CP and trace preserving (TP) at every time by a closed-form certificate (Proposition 1) and it entangles. The listed algebraic, Choi and PPT certificates behind these two facts are formally verified in Lean 4 (Sec. 4.1). The family is not even P-divisible, as Theorem 2 requires. So divisibility cannot be relaxed to complete positivity at all times, given single-time classicality. Divisibility is also not necessary (Remark 3).
4. Section 5: the mechanism is a which-path record written at A and erased at B by trace-preserving conditional maps that are not positive. The register is classical at each single time. Under the stated assumption that the same time-local law continues after a reading, the proposed reading/continuation/measurement rule has negative joint quasi-weights.
5. Section 6: the same choreography placed on a chain of classical cells shaped like Oppenheim's classical-quantum (CQ) master equation [18]. This is a numerical supplement. As built, the chain is not CP, with the stated negative Choi eigenvalues (leading law ce^{-2R_2}), and a dressed two-site member is numerically consistent with CP at the default finite rates; all-times CP remains conjectural.
6. Section 7: what an observation of GIE can and cannot exclude on these assumptions.

1.1 What is claimed, and what is not

Claimed. The defensible contribution is items (i)–(iii): the finite-register generator and gauge formulation, the product-positive refinement for bare inputs, and an explicit all-input, all-times CPTP entangler with one-sided bounded generators. The nearest baselines already establish Markovian and local-round no-entanglement [3, 7, 8], the distinction between total CP and CP-divisibility [17], and restricted correlation-capacity bounds [5]; this paper does not overturn them. Items (iv)–(v) are a numerical supplement and a scope remark. (i) Theorem 1, with its stated hypotheses: finite register, bounded measurable generators, one-sided generators, CP-divisibility. Its proof is a paper proof; it has no formal or human check. Channel-level versions of the conclusion are known [7, 8], as is an analogous Lie–Trotter argument for Markovian bilinear models [19]. Krisnanda *et al.* showed that, for a mediated Hamiltonian and Markovian local environments, zero mediator discord prevents gain of $E_{A:BC}$, and with an initially uncorrelated mediator it also bounds $E_{A:B}$ by its initial value [3]. What we add is the generator-level statement for time-dependent families on a finite hybrid register, including the gauge step (Lemma 2). (ii) Theorem 2, which weakens the hypothesis of Theorem 1 to product-positive divisibility for inputs without ancillas. It is a paper proof, checked at the level of the written proof by one independent model of another provider (Sec. 8); it has no formal or human check. (iii) Theorem 3 and Proposition 1, which together show that the divisibility hypothesis cannot be weakened to complete positivity at all times, given a register that is classical at each single time. The algebraic, Choi and PPT certificates listed in Sec. 4.1 are formally verified in Lean 4; the remaining steps are paper arguments and the decimals are numerical. (iv) The transplant family of Sec. 6, as numerical results only: as built it is not CP, with the stated negative Choi eigenvalues, and the dressed two-site member is numerically consistent with CP at the default rates; all-times CP remains conjectural, even at those rates. That the two stated rate inequalities suffice in general is also a conjecture here; its closed-form certificate is

not included. (v) A scope remark about one sentence of Ref. [18]: the transplant motivates testing a time-dependent, non-divisible extension of the formal generator ingredients under our declared nearest-neighbour mapping. Because exact dressed CP and membership in the named theory are not established, it does not establish an extension or failure of that article’s physical null-result prediction (Sec. 6).

Not claimed. Nothing here is about gravity. The objects are finite lattice toys, tuned to entangle, with no continuum limit and no spacetime. We do not claim that any physical theory of classical gravity has members of this kind, and we know of no physical model that does. The implications of GIE experiments for named candidate theories are exactly as open as before. Theorems 1 and 2 do not cover continuous classical registers or unbounded generators, and Theorem 2 does not cover inputs with local ancillas. We do not claim that P-divisibility, or any divisibility, is necessary. The dressed-member evidence is numerical and covers only the two-site chain, this register family and the default rates; CP of the full dressed family at every time is not certified even there, and the rate region $S1 \wedge S2$ is conjectural.

Readings we concede. Tomizuka and Takeda show that non-Markovian CQ dynamics arise as effective descriptions of a quantum mediator coupled to unobserved degrees of freedom [17]. Tomizuka–Takeda motivates an analogous hidden-quantum interpretation of our constructions; we do not construct a compatible quantum-environment dilation of these finite families here, and nothing in this paper contradicts that interpretation. Corollary 3 is consistent with it: no entangling member can be obtained from a CP-divisible, one-sided dynamics on a larger finite classical register. Ganardi *et al.* read non-decomposability as evidence of a nonclassical interaction [5]. Corollary 1 excludes factors that preserve the classical-register algebra. Snapshot classicality of the given family does not constrain hidden quantum factor states or the supremum over all theory-allowed states in Ganardi *et al.*; unrestricted quantum-channel decomposability of this family is undetermined here (Sec. 7).

2 Setting and definitions

Let $H = H_A \otimes H_B = \mathbb{C}^2 \otimes \mathbb{C}^2$ and let S be a finite register alphabet. The hybrid algebra is $\mathfrak{A}_S = \bigoplus_{s \in S} B(H)$, with elements $X = (X_s)_{s \in S}$. A hybrid state is $\varrho = (\varrho_s)$ with $\varrho_s \geq 0$ and $\sum_s \text{tr } \varrho_s = 1$, equivalently $\varrho = \sum_s |s\rangle\langle s| \otimes \varrho_s$. The register is classical: it carries the distribution $p(s) = \text{tr } \varrho_s$ and has zero quantum discord with AB . A linear map Φ on $\mathfrak{A}_S$ has blocks $\Phi_{s's} : B(H) \rightarrow B(H)$, with $(\Phi X)_{s'} = \sum_s \Phi_{s's}(X_s)$. For a map ϕ on $B(\mathbb{C}^d)$ we write $J(\phi) = \sum_{ij} \phi(|i\rangle\langle j|) \otimes |i\rangle\langle j|$ for its Choi matrix [20] and $|\Omega\rangle = \sum_i |ii\rangle$ (so $\langle \Omega | \Omega \rangle = d$).

Definition 1 (hybrid CP and TP). Φ is hybrid CP if $J(\Phi_{s's}) \geq 0$ for every block (by Choi’s theorem [20], equivalently every block $\Phi_{s's}$ is CP), and TP if $\sum_{s'} \text{tr } \Phi_{s's}(X) = \text{tr } X$ for all s and X .

Hybrid CP in this blockwise sense is equivalent to complete positivity of Φ as a map on the C^* -algebra $\mathfrak{A}_S$ (paper proof: an element of $M_n(\mathfrak{A}_S)$ is positive iff each register component is, and $\text{id}_n \otimes \Phi$ acts on component s' as $\sum_s \text{id}_n \otimes \Phi_{s's}$; so blockwise CP implies CP, and inputs supported on a single component s give the converse). A hybrid CPTP map sends hybrid states to hybrid states, so the register stays classical at the output time.

Definition 2 (hybrid dynamics; mediated generator). A family $(\Phi_t)_{t \in [0, T]}$ is absolutely continuous, has $\Phi_0 = \text{id}$, and solves $\dot{\Phi}_t = L_t \Phi_t$ for almost every t . (Without absolute continuity the a.e. equation

does not determine Φ : a family that jumps at a window boundary satisfies it on each open window.) It is *mediated* if $L_t = L_t^A + L_t^B$ with

$$(L_t^A)_{s's} = \ell_{t;s's}^A \otimes \text{id}_B, \quad (L_t^B)_{s's} = \text{id}_A \otimes \ell_{t;s's}^B,$$

where $t \mapsto L_t^A, L_t^B$ are measurable and locally bounded. No term acts nontrivially on both A and B . Register jumps can appear in either part.

If every Φ_t is hybrid CPTP, the register is classical at each single time: the family is *single-time classical* (“snapshot-classical”). This is a statement about the one-time states $\Phi_t(\varrho)$ only. It does not say that the register can be read at an intermediate time s , with the state continuing under $\Phi_{t,s}$, while all joint probabilities of that reading and later measurements stay nonnegative; for a block-diagonal register, under that same post-reading continuation assumption, the $(r' \leftarrow r)$ block of $\Phi_{t,s}$ is the proposed conditional map given readings r at s and r' at t , so that multi-time property is a condition on $\Phi_{t,s}$ (Sec. 5). Bounded generators make every Φ_t invertible (Liouville’s formula), so the intermediate maps $\Phi_{t,s} := \Phi_t \Phi_s^{-1}$ are defined.

Definition 3 (CP-divisibility [14, 15]). (Φ_t) is CP-divisible if $\Phi_{t,s}$ is hybrid CPTP for all $0 \leq s \leq t \leq T$.

Definition 4 (blockwise separability; entangling). Let $\mathcal{S}_{\text{sep}} \subset \mathfrak{A}_S$ be the set of X whose every block X_s lies in the closed convex cone generated by $\{\alpha \otimes \beta : \alpha, \beta \geq 0\}$. The family is *entangling* if for some s_0 , product state $\rho_A \otimes \rho_B$ and time t , the reduced state $\sum_s [\Phi_t(|s_0\rangle\langle s_0| \otimes \rho_A \otimes \rho_B)]_s$ is entangled.

For two qubits we detect entanglement with the partial transpose [21, 22] and quantify it by the negativity $N(\rho) = \sum_{\lambda < 0} |\lambda|$ over the eigenvalues of ρ^{T_B} [23].

Definition 2 is mediation at the level of generators. Di Biagio’s mediation (Definition 2 and Eq. (3) of Ref. [8]) is at the level of channels, as a product of rounds $(\mathbb{K}_A \otimes T_{BG}^{(n)}) \circ (T_{AG}^{(n)} \otimes \mathbb{K}_B)$. The channel sequences of Corollary 1 of Ref. [7] are of the same kind. The single-round decomposability of Ganardi *et al.* is the special case $\lambda = \lambda_{BM} \circ \lambda_{AM}$ [5].

3 Sufficiency and tightness

Theorem 1 (CP-divisibility suffices). *Let (Φ_t) be a mediated hybrid dynamics (Definition 2) on a finite register. If it is CP-divisible, then $\Phi_t(\mathcal{S}_{\text{sep}}) \subseteq \mathcal{S}_{\text{sep}}$ for every t . In particular it is not entangling.*

The proof uses four lemmas. Write $Q_X = \mathbb{K} - |\Omega_X\rangle\langle\Omega_X|/d_X$ for the projector onto $|\Omega_X\rangle^\perp$ in $H_X \otimes H_X$.

Lemma 1 (generator conditions). *If (Φ_t) is CP-divisible then, for almost every u : (i) $J(L_{u;s's}) \geq 0$ for $s' \neq s$; (ii) $Q J(L_{u;ss}) Q \geq 0$, where Q projects onto $|\Omega\rangle^\perp$ in $H \otimes H$.*

Proof. At every Lebesgue point u of L , $h^{-1}(\Phi_{u+h,u} - \text{id}) \rightarrow L_u$ as $h \downarrow 0$. The map $\Phi_{u+h,u}$ is hybrid CP, so $J((\Phi_{u+h,u})_{s's}) \geq 0$. For $s' \neq s$ the identity contributes nothing to the (s', s) block, and (i) follows in the limit. For $s' = s$, $J(\text{id}) = |\Omega\rangle\langle\Omega|$ vanishes on $|\Omega\rangle^\perp$, and (ii) follows. $\square$

Lemma 2 (gauge). *Let C_A, C_B be Hermitian on $H_A \otimes H_A$ and $H_B \otimes H_B$, and let $\Omega_X = |\Omega_X\rangle\langle\Omega_X|$. Regroup tensor factors as $(A_{\text{out}} A_{\text{in}})(B_{\text{out}} B_{\text{in}})$. Then*

- (a) $C_A \otimes \Omega_B + \Omega_A \otimes C_B \geq 0$ if and only if there is $\theta \in \mathbb{R}$ with $C_A + \theta \Omega_A \geq 0$ and $C_B - \theta \Omega_B \geq 0$;
(b) with Q the projector onto $(|\Omega_A\rangle \otimes |\Omega_B\rangle)^\perp$, $Q(C_A \otimes \Omega_B + \Omega_A \otimes C_B)Q \geq 0$ if and only if $Q_A C_A Q_A \geq 0$ and $Q_B C_B Q_B \geq 0$.

Proof. Let $e_X = |\Omega_X\rangle/\sqrt{d_X}$ and write $C_X = \begin{pmatrix} \alpha_X & a_X^\dagger \\ a_X & D_X \end{pmatrix}$ in the splitting $\mathbb{C}e_X \oplus e_X^\perp$, so that $\Omega_X = d_X e_X e_X^\dagger$. The operator $M = C_A \otimes \Omega_B + \Omega_A \otimes C_B$ vanishes on $e_A^\perp \otimes e_B^\perp$ and on its complement $\mathbb{C}(e_A \otimes e_B) \oplus (e_A^\perp \otimes e_B) \oplus (e_A \otimes e_B^\perp)$ has the arrowhead form

$$M = \begin{pmatrix} d_B \alpha_A + d_A \alpha_B & d_B a_A^\dagger & d_A a_B^\dagger \\ d_B a_A & d_B D_A & 0 \\ d_A a_B & 0 & d_A D_B \end{pmatrix}.$$

By the Schur complement, $M \geq 0$ iff $D_A, D_B \geq 0$, $a_X \in \text{ran } D_X$, and $d_B \alpha_A + d_A \alpha_B \geq d_B a_A^\dagger D_A^+ a_A + d_A a_B^\dagger D_B^+ a_B$. Likewise $C_A + \theta \Omega_A \geq 0$ iff $D_A \geq 0$, $a_A \in \text{ran } D_A$ and $\theta \geq \theta_- := (a_A^\dagger D_A^+ a_A - \alpha_A)/d_A$. And $C_B - \theta \Omega_B \geq 0$ iff $D_B \geq 0$, $a_B \in \text{ran } D_B$ and $\theta \leq \theta_+ := (\alpha_B - a_B^\dagger D_B^+ a_B)/d_B$. The interval $[\theta_-, \theta_+]$ is nonempty exactly when the Schur inequality holds. This proves (a). For (b), $QM = d_B D_A \oplus d_A D_B$ on $(e_A^\perp \otimes e_B) \oplus (e_A \otimes e_B^\perp)$, and zero elsewhere. $\square$

Lemma 3 (hybrid GKLS generators). *Let K be a Hermiticity-preserving map on $\mathfrak{A}_S$ with $J(K_{s's}) \geq 0$ for $s' \neq s$, $QJ(K_{ss})Q \geq 0$ for all s , and $\sum_{s'} \text{tr } K_{s's}(X) = 0$. Then $e^{\tau K}$ is hybrid CPTP for every $\tau \geq 0$.*

Proof. Split $K = D + R$ into its block-diagonal part D and its off-diagonal part R . R is hybrid CP. Each K_{ss} is Hermiticity preserving and conditionally completely positive on $B(H)$, so it generates CP maps [24, 25, 26], and $e^{\tau D}$ is hybrid CP. Indeed, subtract the CP map with Choi matrix $QJ(K_{ss})Q \geq 0$. The remaining Hermitian Choi matrix has zero Q -compression and is $|\Omega\rangle\langle v| + |v\rangle\langle \Omega|$ for some vector v , hence represents $X \mapsto A_s X + X A_s^\dagger$ for some operator A_s . The drift semigroup $X \mapsto e^{\tau A_s} X e^{\tau A_s^\dagger}$ is CP, and its Dyson perturbation by the CP part is CP; no trace assumption is used. The Dyson series

$$e^{\tau(D+R)} = \sum_{n \geq 0} \int_{0 \leq \sigma_1 \leq \dots \leq \sigma_n \leq \tau} e^{(\tau - \sigma_n)D} R \dots R e^{\sigma_1 D} d\sigma$$

converges in norm and each term is CP, so the sum is CP. Trace preservation follows from $\sum_{s'} \text{tr } K_{s's} = 0$. $\square$

Lemma 4 (one-sided maps preserve $\mathcal{S}_{\text{sep}}$). *If Ψ is hybrid CP with blocks $\Psi_{s's} = \psi_{s's} \otimes \text{id}_B$ (or $\text{id}_A \otimes \psi_{s's}$), then $\Psi(\mathcal{S}_{\text{sep}}) \subseteq \mathcal{S}_{\text{sep}}$.*

Proof. $\psi_{s's}$ is CP because $\psi_{s's} \otimes \text{id}_B$ is. A block $\sum_k \alpha_k \otimes \beta_k$ maps to $\sum_k \psi_{s's}(\alpha_k) \otimes \beta_k$ with $\psi_{s's}(\alpha_k) \geq 0$, and sums of such terms stay in the cone. (Lemma 1 of Ref. [7] is the general C^* -algebraic version.) $\square$

Proof of Theorem 1. Fix u as in Lemma 1. We may take ℓ_u^A and ℓ_u^B Hermiticity preserving by replacing each with its Hermitian part. For $s' \neq s$, $J(L_{u;s's}) = J(\ell_{u;s's}^A) \otimes \Omega_B + \Omega_A \otimes J(\ell_{u;s's}^B) \geq 0$ after regrouping. Lemma 2(a) gives $\theta_{u;s's}$, and we replace $\ell_{u;s's}^A \rightarrow \ell_{u;s's}^A + \theta \text{id}_A$ and $\ell_{u;s's}^B \rightarrow \ell_{u;s's}^B - \theta \text{id}_B$. We choose θ as the point of $[\theta_-, \theta_+]$ closest to 0. This choice is Borel in u and satisfies $|\theta| \leq \max(\|J(\ell^A)\|/d_A, \|J(\ell^B)\|/d_B)$, so the new parts remain measurable and locally

bounded. For $s' = s$, Lemma 2(b) shows that each part is conditionally CP, and adding multiples of the identity does not change this. Trace preservation of L_u gives $\sum_{s'} \text{tr} \ell_{u;s's}^A(\cdot) = c_s \text{tr}(\cdot)$ and $\sum_{s'} \text{tr} \ell_{u;s's}^B(\cdot) = -c_s \text{tr}(\cdot)$ for some constants c_s , removed by the diagonal shift $\ell_{u;ss}^A \rightarrow \ell_{u;ss}^A - c_s \text{id}$, $\ell_{u;ss}^B \rightarrow \ell_{u;ss}^B + c_s \text{id}$. So for almost every u , $L_u = \tilde{L}_u^A + \tilde{L}_u^B$ with both parts one-sided and satisfying Lemma 3.

Partition $[0, t]$ uniformly into n intervals and replace L by its interval averages. The averages converge to L in L^1 , so the corresponding propagators $\Phi_t^{(n)}$ converge to Φ_t by Grönwall's inequality. The generators satisfying Lemma 3 form a closed convex cone, so each average splits into one-sided parts of the same kind. By the Lie product formula each factor $e^{\delta(\tilde{L}^A + \tilde{L}^B)}$ is a limit of products of one-sided hybrid CPTP maps. These preserve $\mathcal{S}_{\text{sep}}$ by Lemma 4, and $\mathcal{S}_{\text{sep}}$ is closed, so $\Phi_t(\mathcal{S}_{\text{sep}}) \subseteq \mathcal{S}_{\text{sep}}$. A product input $|s_0\rangle\langle s_0| \otimes \rho_A \otimes \rho_B$ lies in $\mathcal{S}_{\text{sep}}$, and a sum of separable blocks is separable. $\square$

Remark 1 (status). The proof above was assembled by the drafting agent from the project's refereed-once record of this result and from Lemma 2, which the drafting agent re-derived in closed form and tested on 300 random instances (`drafting_checks.py`, part 3). An independent full-paper review by a model of another provider judged the proof sound (Sec. 8); no human has checked it. Local boundedness is used in the proof. Families with singular or unbounded generators, including an instantaneous flush, are not covered.

Remark 2 (tightness). Theorem 3 below exhibits a mediated, single-time classical family, CP and TP at every time, that entangles. The divisibility hypothesis therefore cannot be replaced by complete positivity at all times together with single-time classicality.

Remark 3 (divisibility is not necessary). Let L_0 be any time-independent mediated generator satisfying Lemma 3 with at least one nonzero off-diagonal register block, and let $f \geq 0$ be smooth with $f(0) = 0$ and not monotone, for example $f(t) = \sin t$ on $[0, \pi]$. Then $\Phi_t = e^{f(t)L_0}$ is mediated with generator $f'(t)L_0$ and is CPTP at every t . It is not CP-divisible: where f decreases, $\Phi_{t,s} = e^{-(f(s)-f(t))L_0}$ has, in any register block (s', s) with $s' \neq s$ where $(L_0)_{s's} \neq 0$, the value $-(f(s) - f(t))(L_0)_{s's} + O((f(s) - f(t))^2)$. Since $(L_0)_{s's}$ is CP and nonzero, this block is not CP for small positive $f(s) - f(t)$. It is not even product-positive in the sense of Definition 5: product projectors span $B(H)$, so some $P_a \otimes P_b$ has a nonzero positive image under $(L_0)_{s's}$, product vectors span H , so some $x \otimes y$ gives that image a positive expectation, and the leading term then makes the corresponding value of Eq. (1) negative. It never entangles, because Φ_t coincides with the CP-divisible family $e^{\tau L_0}$ at $\tau = f(t)$ and Theorem 1 applies. So neither CP-divisibility nor the weaker hypothesis of Theorem 2 is necessary. The example is elementary and was written by the drafting agent. Memoryful classical-gravity models that do not entangle by construction are in print, for example Ref. [27], but their divisibility has not been analysed.

3.1 Positive divisibility suffices for inputs without ancillas

Theorem 1 assumes that every intermediate map is completely positive. For inputs without ancillas, positivity is enough, and a weaker, product-tested positivity already suffices. Throughout this subsection $(\Phi_t)_{t \in [0, T]}$ is a mediated hybrid dynamics (Definition 2) on a finite register S , with Φ_t Hermiticity preserving, and $\Phi_{t,s} = \Phi_t \Phi_s^{-1}$. We write $P_v = |v\rangle\langle v|$ for unit vectors v .

Definition 5 (product-positive divisibility). (Φ_t) is *product-positive divisible* if for all $0 \leq s \leq t \leq T$, all $z, z' \in S$ and all unit $a, x \in H_A, b, y \in H_B$,

$$\langle x \otimes y | (\Phi_{t,s})_{z'z} (P_a \otimes P_b) | x \otimes y \rangle \geq 0. \quad (1)$$

It is *P-divisible* if every $\Phi_{t,s}$ is a positive map on $\mathfrak{A}_S$, i.e. every block $(\Phi_{t,s})_{z'z}$ is a positive map on $B(H_A \otimes H_B)$. P-divisibility implies product-positive divisibility, and CP-divisibility implies P-divisibility.

Operationally, Eq. (1) is the statement that a protocol “read z at time s , reset A to a and B to b , read z' at time t , measure x on A and y on B ” has a nonnegative joint weight, when the time-local law is assumed to govern the state after the reading at s . It is the time-dependent form of a condition for autonomous dynamics in the project record, with $\Phi_{t,s}$ in place of $e^{\tau L}$.

Theorem 2 (product-positive divisibility suffices for bare inputs). *If a mediated hybrid dynamics on a finite register is product-positive divisible (in particular, if it is P-divisible), then $\Phi_t(\mathcal{S}_{\text{sep}}) \subseteq \mathcal{S}_{\text{sep}}$ for every t . In particular it is not entangling: from any product input $|z_0\rangle\langle z_0| \otimes \rho_A \otimes \rho_B$ every register block of Φ_t , and their sum, is separable. Mixed (unattributed) register jumps and time-dependent generators are allowed.*

The proof is in Appendix A. It follows the route of Theorem 1 with positive maps in place of CP maps: generator inequalities at Lebesgue points (Lemma 5), an arrowhead lemma for general dimension (Lemma 6), a rank-one form of the gauge of Lemma 2 that splits every averaged generator into one-sided halves with positive jumps and cross-positive dwells (Lemma 7), Kossakowski’s criterion for positive semigroups [28], and a Lie–Trotter limit. Trace preservation is not used, and the gauge is chosen per averaging interval, so no measurable selection in time is needed. In the project record the result is Lemma B05.

Remark 4 (scope and tightness). (i) *Complete positivity, or positivity, at all times is not enough.* The family of Theorem 3 is CP at all times and entangles; it violates Eq. (1): on $|5\rangle\langle 5| \otimes P_0 \otimes P_0$ the $(\dagger \leftarrow 5)$ block of $\Phi_{t,s}$, $2 \leq s < t \leq 3$, gives $(1 - e^{-\kappa(t-s)})P_0 \otimes Y_5$, and $x = |0\rangle$, y the minimal eigenvector of Y_5 give $-(1 - e^{-\kappa(t-s)})\Lambda_\star < 0$. (ii) *Bare inputs only.* The one-sided halves produced by Lemma 7 are positive, not necessarily CP, so the proof does not show that separability is preserved when local ancillas A' , B' are attached. Applying Theorem 2 to the ancilla-extended family $\Phi_t \otimes \text{id}_{A'B'}$ requires its hypothesis for that family across the cut $AA':BB'$, a stronger condition on (Φ_t) . CP-divisibility of (Φ_t) suffices with ancillas (Theorem 1), and, for trace-preserving families, P-divisibility of the extended family, with $\dim H_{A'} \geq \dim H_A$ and $\dim H_{B'} \geq \dim H_B$, is equivalent to CP-divisibility of (Φ_t) . We do not know whether a P-divisible mediated family can entangle when local ancillas are present. (iii) *The hypothesis is strictly weaker than CP-divisibility, even within P-divisibility:* Example 1 is mediated, CPTP at every time, P-divisible, and not CP-divisible, so Theorem 1 does not apply to it and Theorem 2 does. Product-positive divisibility is strictly weaker again: $\Phi_t = e^{t(\mathbb{T} - \text{id})} \otimes \text{id}_B$ with $\mathbb{T}$ the transpose on H_A satisfies Eq. (1) but Φ_t is not positive on $B(H_A \otimes H_B)$ for $t > 0$. (iv) *Where the P/CP gap lives.* At a Lebesgue point of the generator, if it acts on one side only, say $L_u = L_u^A$, and $d_B \geq d_A$, then P-divisibility forces the halves to be CP (jumps) and conditionally CP (dwells), because $\psi \otimes \text{id}_B$ positive on $B(H_A \otimes H_B)$ means ψ is d_B -positive, hence CP. So on two qubits a P-divisible, not CP-divisible mediated family needs both halves active at the same instant, as in Example 1, where noise on B pays for a non-CP dwell on A .

Example 1 (P-divisible, CPTP at all times, not CP-divisible). Two qubits, a one-element register. Let $\Delta(X) = \text{tr}(X)\mathbb{I}/2 - X$ and let ℓ_g be the Pauli generator $\ell_g(X) = \sigma_1 X \sigma_1 + \sigma_2 X \sigma_2 - g \sigma_3 X \sigma_3 - (2 - g)X$ with $0 < g \leq 1$. On $[0, 1]$ the generator is $\Delta \otimes \text{id}$; on $[1, 2]$ it is $G = \ell_g \otimes \text{id} + \text{id} \otimes \kappa \Delta$ with $\kappa \geq 4g$.

- (a) G is cross-positive on $B(\mathbb{C}^2 \otimes \mathbb{C}^2)$, so (Φ_t) is P-divisible.

- (b) For $g = 1$: $\Phi_{t,s}$, $1 \leq s < t \leq 2$, is not CP; its Choi matrix has minimum eigenvalue $-\frac{1}{4}(1 - e^{-4\tau})(1 + 3e^{-\kappa\tau})$, $\tau = t - s$ (for $\kappa = 4$).
- (c) For $g = 1$, Φ_t is CPTP for every $t \in [0, 2]$.

The proof is in Appendix A.

4 An explicit counterexample

Construction. The register is $S = \{\emptyset, 1, \dots, 6, \dagger\}$: blank, six records and a sink. Let $K_m = \text{diag}(a_m, b_m)$ on A , with $a_m = \sqrt{p_m}$, $b_m = \sqrt{q_m} e^{i\vartheta_m}$, and

$$p = (0.4, 0.2, 0.1, 0.2, 0.05, 0.05), \quad q = (0.1, 0.2, 0.4, 0.2, 0.05, 0.05), \quad \vartheta = (0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 0, 0).$$

Then $\sum_m K_m^\dagger K_m = \mathbb{K}$ and $\gamma := \sum_m a_m \bar{b}_m = 1/10$. Set $\mu = \gamma = 1/10$, $\nu = 1/20$ and $w = 1 - \mu - \nu = 17/20$, and define operators on H_B :

$$T_{00} = (w + \mu)|0\rangle\langle 0| + \frac{\nu}{2}\mathbb{K}, \quad T_{11} = w|1\rangle\langle 1| + \mu|0\rangle\langle 0| + \frac{\nu}{2}\mathbb{K}, \quad T_{01} = T_{10}^\dagger = w|0\rangle\langle 1| + \mu|0\rangle\langle 0|.$$

Let $Y_1, \dots, Y_6$ be Hermitian 2×2 matrices with $\text{tr } Y_m = 1$ that solve the moment equations

$$\sum_m |a_m|^2 Y_m = T_{00}, \quad \sum_m a_m \bar{b}_m Y_m = T_{01}, \quad \sum_m |b_m|^2 Y_m = T_{11}. \quad (2)$$

We use the solution

$$Y_1 = \begin{pmatrix} \frac{1283}{492} & -\frac{85}{164} \\ -\frac{85}{164} & -\frac{791}{492} \end{pmatrix}, \quad Y_2 = \bar{Y}_4 = \begin{pmatrix} \frac{259}{246} & -\frac{17}{41} + \frac{17i}{16} \\ -\frac{17}{41} - \frac{17i}{16} & -\frac{13}{246} \end{pmatrix}, \quad Y_3 = \begin{pmatrix} -\frac{37}{164} & -\frac{85}{164} \\ -\frac{85}{164} & \frac{201}{164} \end{pmatrix}, \quad Y_5 = Y_6 = \begin{pmatrix} -\frac{14}{3} & \frac{17}{4} \\ \frac{17}{4} & \frac{17}{3} \end{pmatrix}.$$

This is the solution used by the original scripts (checked to 2×10^{-14} in floating arithmetic in `drafting_ycheck.py`; the exact moment identities are independently certified, Sec. 4.1). Every Y_m has a negative eigenvalue: $\lambda_{\min}(Y_m) \approx -1.67, -0.77, -0.39, -0.77, -6.19, -6.19$. The most negative is $\lambda_{\min}(Y_5) = \frac{1}{2} - \frac{\sqrt{6445}}{12} = -\Lambda_\star$ with $\Lambda_\star = (\sqrt{6445} - 6)/12 \approx 6.190063$. The conditional maps $\mathcal{G}_m(X) = \text{tr}(X) Y_m$ on B are trace preserving and not positive: each discards B and replaces it by the non-positive matrix Y_m . Write $\mathcal{K}_m(X) = (K_m \otimes \mathbb{K})X(K_m \otimes \mathbb{K})^\dagger$ and $\tilde{\mathcal{G}}_m = \text{id}_A \otimes \mathcal{G}_m$.

The dynamics has three unit windows with rates $R_2 = 14$, $R = 12$, $\kappa = 6$. Nonzero generator blocks are:

- W_1 , $t \in [0, 1]$, flush (register only): $(\emptyset \leftarrow m) = R_2 \text{id}$, $(m \leftarrow m) = -R_2 \text{id}$;
- W_2 , $t \in [1, 2]$, write (A and register): $(m \leftarrow \emptyset) = R \mathcal{K}_m$, $(\emptyset \leftarrow \emptyset) = -R \text{id}$;
- W_3 , $t \in [2, 3]$, erase (B and register): $(\dagger \leftarrow m) = \kappa \tilde{\mathcal{G}}_m$, $(m \leftarrow m) = -\kappa \text{id}$.

Theorem 3 (explicit counterexample). *Let G_k be the generator of window W_k and r_k its rate, and let Φ_t be the time-ordered product of the window propagators $E_k(u) = \text{id} + r_k^{-1}(1 - e^{-r_k u})G_k$, where u is the time elapsed in window k . This is the absolutely continuous solution of Definition 2 with $\Phi_0 = \text{id}$ (uniqueness is the standard linear-ODE fact). The family $(\Phi_t)_{t \in [0, 3]}$ has the following properties.*

- (a) *The generator is piecewise constant and mediated: W_1 acts on the register only, the W_2 blocks are $\ell_m \otimes \text{id}_B$ with $\ell_m(\alpha) = K_m \alpha K_m^\dagger$ (or $-R \text{id}$), and the W_3 blocks are $\text{id}_A \otimes \mathcal{G}_m$ (or $-\kappa \text{id}$).*

- (b) Φ_t is hybrid CPTP for every $t \in [0, 3]$. Hence the register is classical at each single time. If the same time-local law is assumed to govern the state after an intermediate reading of the register, it is not classical at two times jointly with measurements on A and B (Sec. 5).
- (c) (Φ_t) is not CP-divisible, and not even P-divisible or product-positive divisible, so neither Theorem 1 nor Theorem 2 applies (Remark 4(i)). For $2 \leq s < t \leq 3$ the block $(\dagger \leftarrow 5)$ of $\Phi_{t,s}$ has minimum Choi eigenvalue exactly $-2(1 - e^{-\kappa(t-s)})\Lambda_*$, numerically -3.2087 for $t - s = 0.05$ and -12.3494 for $t - s = 1$.
- (d) From $\varrho_0 = |\emptyset\rangle\langle\emptyset| \otimes |+\rangle\langle+| \otimes |+\rangle\langle+|$, the $A:B$ state at $t = 3$ is

$$\tau = e^{-R}|++\rangle\langle++| + (1 - e^{-R}) \left[e^{-\kappa} \mathcal{D}_\gamma(|+\rangle\langle+| \otimes |+\rangle\langle+| + (1 - e^{-\kappa})(w\Phi^+ + \mu|+\rangle\langle+| \otimes |0\rangle\langle 0| + \frac{\nu}{4}\mathbb{K}) \right], \quad (3)$$

where $\mathcal{D}_\gamma$ multiplies off-diagonal entries by γ and Φ^+ is the maximally entangled projector. Its negativity is $N(\tau) = 0.388021 \dots$ (numerical). The singlet $|\psi^-\rangle$ gives the hand-checkable witness

$$\langle\psi^-|\tau^{TB}|\psi^-\rangle = -\frac{1}{80}(1 - e^{-R})(31 - 49e^{-\kappa}) \approx -0.3860 < 0. \quad (4)$$

For $t \leq 2$ the $A:B$ state is a mixture of two product states.

Proposition 1 (all-times certificate; “Lemma E” of the project record). Let $\mathcal{T} = \sum_m \tilde{\mathcal{G}}_m \mathcal{K}_m$. If

$$\nu(1 - e^{-R_2})(1 - e^{-R}) \geq 4\Lambda_* e^{-R_2}, \quad (5)$$

then Φ_t is hybrid CPTP for every $t \in [0, 3]$. At the stated rates Eq. (5) holds; numerically, the left side exceeds the right side by a factor 2428.5.

Proof. Each window generator G with rate r satisfies $G^2 = -rG$, so $e^{uG} = \text{id} + r^{-1}(1 - e^{-ru})G$. On $[0, 2]$ every block of Φ_t is therefore a nonnegative combination of id and the $\mathcal{K}_m$, hence CP. Let $M = \Phi_2$. Its nonzero blocks are $M_{\emptyset\emptyset} = e^{-R}\text{id}$, $M_{m\emptyset} = (1 - e^{-R})\mathcal{K}_m$, $M_{\emptyset m} = (1 - e^{-R_2})e^{-R}\text{id}$, $M_{m'm} = (1 - e^{-R_2})(1 - e^{-R})\mathcal{K}_{m'} + \delta_{m'm}e^{-R_2}\text{id}$ and $M_{\dagger\dagger} = \text{id}$. For $t = 2 + u$, $u \in [0, 1]$, the blocks of Φ_t are $M_{\emptyset s}$, $e^{-\kappa u}M_{ms}$ and $\delta_{s\dagger}\text{id} + (1 - e^{-\kappa u})N_s$, where

$$N_{\emptyset} = (1 - e^{-R})\mathcal{T}, \quad N_m = (1 - e^{-R_2})(1 - e^{-R})\mathcal{T} + e^{-R_2}\tilde{\mathcal{G}}_m, \quad N_{\dagger} = 0.$$

It therefore suffices that $\mathcal{T}$ and every N_m are CP. Let $|\tilde{0}\rangle = |00\rangle$ and $|\tilde{1}\rangle = |11\rangle$ in $A_{\text{out}}A_{\text{in}}$. The Choi matrix of the A -only map ℓ_m is $v_m v_m^\dagger$ with $v_m = a_m|\tilde{0}\rangle + b_m|\tilde{1}\rangle$, so that $J(\mathcal{K}_m) = v_m v_m^\dagger \otimes |\Omega_B\rangle\langle\Omega_B|$ after regrouping. Using Eq. (2),

$$J(\mathcal{T}) = X \otimes \mathbb{K}_{B_{\text{in}}}, \quad X = \sum_{ij} |\tilde{i}\rangle\langle\tilde{j}| \otimes T_{ij} = 2w|\tilde{\Phi}\rangle\langle\tilde{\Phi}| + 2\mu|\tilde{\dagger}\rangle\langle\tilde{\dagger}| \otimes |0\rangle\langle 0| + \frac{\nu}{2}\mathbb{K} \geq \frac{\nu}{2}\mathbb{K}$$

on $\text{span}\{|\tilde{0}\rangle, |\tilde{1}\rangle\} \otimes H_{B_{\text{out}}}$. Here $|\tilde{\Phi}\rangle = (|\tilde{00}\rangle + |\tilde{11}\rangle)/\sqrt{2}$ and $|\tilde{\dagger}\rangle = (|\tilde{0}\rangle + |\tilde{1}\rangle)/\sqrt{2}$, and $J(\mathcal{T})$ vanishes off this subspace. On the same subspace $J(\tilde{\mathcal{G}}_m) = 2|\tilde{\dagger}\rangle\langle\tilde{\dagger}| \otimes Y_m \otimes \mathbb{K}_{B_{\text{in}}} \geq -2\Lambda_*\mathbb{K}$, and it also vanishes off it. Hence $J(N_m) \geq [(1 - e^{-R_2})(1 - e^{-R})\frac{\nu}{2} - 2\Lambda_*e^{-R_2}]\mathbb{K} \geq 0$ under Eq. (5). Every generator annihilates the trace because each $\mathcal{G}_m$ is TP. $\square$

Proof of Theorem 3. (a) Read off the window blocks: $\mathcal{K}_m = \ell_m \otimes \text{id}_B$ and $\tilde{\mathcal{G}}_m = \text{id}_A \otimes \mathcal{G}_m$ as maps on all of $B(H)$, so each window generator is $L^A + L^B$ with one of the two parts zero. (b) This is Proposition 1. (c) For $2 \leq s < t \leq 3$, $\Phi_{t,s} = e^{(t-s)G_3}$, whose $(\dagger \leftarrow 5)$ block is $(1 - e^{-\kappa(t-s)})\tilde{\mathcal{G}}_5$ with Choi matrix $2(1 - e^{-\kappa(t-s)})|\tilde{\dagger}\rangle\langle\tilde{\dagger}| \otimes Y_5 \otimes \mathbb{K}$. Applied to the hybrid state $|5\rangle\langle 5| \otimes |00\rangle\langle 00|$ it gives the

sink block $(1 - e^{-\kappa(t-s)})|0\rangle\langle 0| \otimes Y_5$, which is not positive semidefinite, so $\Phi_{t,s}$ is not even positive. (d) From the blank register, W_1 acts trivially. After W_2 the register is blank with weight e^{-R} or holds m with $\mathcal{K}_m$ applied. After W_3 each record has been erased with weight $1 - e^{-\kappa}$. Summing over the register gives $\tau = e^{-R}\rho_0 + (1 - e^{-R})[e^{-\kappa} \sum_m \mathcal{K}_m(\rho_0) + (1 - e^{-\kappa})\mathcal{T}(\rho_0)]$. Here $\sum_m \mathcal{K}_m$ is the dephasing $\mathcal{D}_\gamma$ on A , and $\mathcal{T}(\rho_A \otimes \rho_B) = \sum_{ij} \langle i|\rho_A|j\rangle \langle i|\langle j| \otimes T_{ij}$, which yields Eq. (3). Equation (4) follows from $\langle \psi^- | (\Phi^+)^{T_B} | \psi^- \rangle = -\frac{1}{2}$, $\langle \psi^- | (|+\rangle\langle +| \otimes |0\rangle\langle 0|)^{T_B} | \psi^- \rangle = \frac{1}{4}$, $\langle \psi^- | (\mathbb{K}/4)^{T_B} | \psi^- \rangle = \frac{1}{4}$, and $\langle \psi^- | (\mathcal{D}_\gamma(|+\rangle\langle +|) \otimes |+\rangle\langle +|)^{T_B} | \psi^- \rangle = \frac{1-\gamma}{4}$. For $t \leq 2$ only W_1 and W_2 have acted, so for a product input $\rho_0 = \rho_A \otimes \rho_B$ the $A:B$ state is $e^{-R \max(t-1,0)}\rho_0 + (1 - e^{-R \max(t-1,0)})\mathcal{D}_\gamma(\rho_A) \otimes \rho_B$, a mixture of two product states. The value of $N(\tau)$ is a numerical 4×4 eigenvalue computation. $\square$

Corollary 1 (no round mediation). *The family of Theorem 3 admits no decomposition, at any time t at which it entangles, into a finite or countable sequence of one-sided channels on the classical-register algebra with a weak* limit in the sense of Corollary 1 of Ref. [7]. So it satisfies mediation at the level of generators and violates it at the level of rounds of classical-register-algebra channels. The corollary does not rule out decompositions whose hidden intermediate mediator states are quantum.*

Proof. Such sequences preserve triseparability [7], and the family entangles. $\square$

Remark 5 (inputs). Entanglement requires coherence of A in the record basis. For $\rho_A = |0\rangle\langle 0|$ the output is a product state. Over 20 random product inputs the endpoint negativity ranged from 0.012 to 0.387 in one sample and from 0.097 to 0.3885 in an independent sample (Sec. 10).

4.1 What is formally verified

A Lean 4 package (papers/qgmine/lean, module QgmCert; Lean v4.34.0, Mathlib 5ed2965) checks the following in exact arithmetic, for the explicit K_m , T_{ij} and $Y_1, \dots, Y_6$ above, with CP defined as a positive semidefinite Choi matrix (Choi's theorem is not formalized). Every listed declaration depends only on the axioms `propext`, `Classical.choice` and `Quot.sound`. Declaration names are in the namespace `QgmCert`.

- The Kraus, γ and moment identities, and $\text{tr } Y_m = 1$ with Y_m Hermitian (`kraus_complete`, `gamma_eq`, `moments`, `Y_trace`, `Y_isHermitian`). So the construction does not depend on the printed Y_m being those of the original scripts.
- $\lambda_{\min}(Y_m)$ in closed form, with $\lambda_{\min}(Y_5) = -\Lambda_\star$ (`Y_minEigs`, `Y5_minEig_eq_neg_Lstar`); each $\mathcal{G}_m$ is TP and not positive, and its Choi matrix is not positive semidefinite (`G_tp`, `G_not_positive`, `choi_G_not_posSemidef`).
- The three window generators satisfy $G^2 = -rG$ (`G1_idem`, `G2_idem`, `G3_idem`), and the closed-form Φ_t has $\Phi_0 = \text{id}$ and solves $\dot{\Phi}_t = L_t \Phi_t$ on each open window (`Emap_zero`, `Phi_hasDerivAt`). Uniqueness of the solution and continuity at the window boundaries are not formalized.
- Φ_t is TP and blockwise CP for every $t \geq 0$ under Eq. (5) at any positive rates, and Eq. (5) holds at $R_2 = 14$, $R = 12$ (`Phi_TP`, `lemmaE_general`, `lemmaE`, `CE_holds`). This is Theorem 3(b) and Proposition 1, with hybrid CP exactly as in Definition 1.
- For $2 \leq s \leq t$, $\Phi_t = e^{(t-s)G_3}\Phi_s$ with Φ_s invertible, and the $(\dagger \leftarrow 5)$ block of $e^{(t-s)G_3}$ has minimum Choi eigenvalue exactly $-2(1 - e^{-\kappa(t-s)})\Lambda_\star < 0$ for $t > s$ (`Phi_split`, `Phi_inverse`, `nondivisible`, `nondivisible_neg`). The identification with Theorem 3(c) composes these lemmas; no single declaration states it.

- The $t = 3$ $A:B$ state from $|\emptyset\rangle\langle\emptyset| \otimes |++\rangle\langle++|$ equals τ of Eq. (3), with the singlet witness of Eq. (4) (`endpoint_eq_tau`, `singlet_witness_normalized`, `witness_negative`).
- At $R = 12$, $\kappa = 6$, τ is not a finite sum $\sum_k \alpha_k \otimes \beta_k$ with $\alpha_k, \beta_k \geq 0$ (`ppt_of_separable`, `tau_entangled`, `endpoint_entangled`). Because the PPT set is closed, $\tau \notin \mathcal{S}_{\text{sep}}$ also follows; that closure step is a paper argument.
- The write and erase maps act one-sidedly on product operators (`Kmap_one_sided`, `Gt_one_sided`) and entrywise on all operators (`Kmap_apply`, `Gt_apply`); the decomposition $L_t = L_t^A + L_t^B$ of Definition 2 is not stated in Lean.

Numerical only, not machine-checked: $N(\tau) = 0.388021$, the witness value -0.3860 , the decimals -3.2087 , -12.3494 , 2428.5 and the decimal $\lambda_{\min}(Y_m)$. Not formalized: the equivalence of blockwise CP with CP on $\mathfrak{A}_S$, the closed-cone form of $\mathcal{S}_{\text{sep}}$, mediation on general operators beyond the entrywise formulas, ODE uniqueness, the statement that for $t \leq 2$ the $A:B$ state is a mixture of two products, and everything outside this section, including Theorem 1, its lemmas, the corollaries and Sec. 6.

5 Mechanism: a which-path record written and then erased

In W_2 the register records m with probability p_m if A is in $|0\rangle$ and q_m if A is in $|1\rangle$. Records 1 and 3 therefore carry partial which-path information about A , and the complex amplitudes $a_m \bar{b}_m$ carry the phase information that the moment equations need. In W_3 the record is erased, moved to the sink, which carries no information. At the same time B is replaced by Y_m . Averaged over records with the A -dependent weights, B ends in T_{ij} conditioned on A 's matrix element (i, j) . The endpoint channel $\mathcal{T}$ is thus, up to the admixtures μ and ν , the coherent copy $|i\rangle \mapsto |i\rangle|i\rangle$ of A 's record basis onto B . It is CP because $X = \sum_m v_m v_m^\dagger \otimes Y_m \geq \frac{\nu}{2} \mathbb{I}$, although no single Y_m is positive.

Three controls follow from the results above. (i) If every conditional is CP ($Y_m \geq 0$), then every window generator satisfies Lemma 3, the family is CP-divisible, and Theorem 1 forbids entanglement. The non-CP erasure is therefore essential. (ii) Exact complete positivity at all times requires that the unflushed residue $e^{-R_2} \tilde{\mathcal{G}}_m$, which reaches the erase window when an input record survives the flush, be dominated. In N_m the residue lands in the same register block as the strictly positive endpoint Choi matrix, which is why Eq. (5) suffices. (iii) Any autonomous version with a bounded generator on a finite register that stays CP is a CP-divisible semigroup and cannot entangle (Theorem 1). The time dependence of the schedule is essential.

Single-time, not multi-time, classicality. The register is classical at each single time (Theorem 3(b)). Under the post-reading continuation law stated below, the proposed two-time weights fail positivity: register readings at several times, together with measurements on A and B , then need not have a nonnegative joint distribution. (We make no claim about the register's own multi-time statistics.) Suppose the time-local law of Definition 2 also governs the state after a reading: read the register at $s = 2$, keep the outcome 5 (probability $(1 - e^{-R})(p_5 + q_5)/2 \approx 0.05$), continue with $\Phi_{3,2}$, and measure B at $t = 3$ in the eigenvector of Y_5 with eigenvalue $-\Lambda_\star$. The joint quasi-probability of reading 5 at s , finding the sink at t and that B outcome is $-\frac{p_5 + q_5}{2}(1 - e^{-R})(1 - e^{-\kappa})\Lambda_\star \approx -0.3087$ (a numerical check by an agent referee, re-run for this revision). More generally, under that continuation law and for a block-diagonal register, the $(r' \leftarrow r)$ block of $\Phi_{t,s}$ is the proposed conditional map on AB given readings r at s and r' at t , so CP-divisibility says exactly that every such two-time conditional map is CP. What Theorem 3 shows is therefore that single-time classicality, one-sided

generators and CP at all times do not by themselves exclude entanglement; the multi-time readability that operational notions of a classical mediator, such as a readable message or a monitorable classical system, typically include is not satisfied here. Multitime classicality and hidden memory of quantum processes are analysed in the process-tensor literature [29, 30, 31]; we do not attribute a specific theorem from it to this construction, and it does not establish or exclude priority for it.

Relation to quasi-probability simulation. Each Y_m is a signed combination of states. So $\mathcal{T}$ is a signed combination of one-way LOCC maps, each consisting of a partial measurement of A , a record sent to B , and a preparation at B . This is the structure exploited by circuit knitting [32, 33], where a nonlocal channel is simulated by a quasi-probabilistic mixture of local operations. Every channel admits such a decomposition, so the decomposition alone is not the point. The point is that here it arises as the endpoint of a continuous-time law with a register that is classical at each single time and CP at every time. A referee may reasonably regard the construction as circuit knitting in continuous time. We have not analysed whether it reduces to a known quasi-probability scheme or what its sampling overhead is. The indivisible stochastic processes of Barandes [34] are a further conceptual neighbour. We have read only their abstract-level framing.

6 Numerical supplement: transplant to a CQ-shaped local-field chain

This section is a numerical and conjectural supplement. None of its statements is used in Secs. 3–5, and the original eight-state register of Sec. 4 already proves the separation claimed in this paper.

6.1 The CQ form and its legality condition

Oppenheim’s CQ master equation [18] evolves a classical-quantum state $\hat{\rho}(z; t)$ through couplings $W^{\alpha\beta}(z|z'; x, y)$ between Lindblad operators $\hat{L}_\alpha(x)$ and local classical fields $z(x)$. Three sentences of that article fix what counts as a legal member for our purposes. The quotations are from the arXiv text; see the note in the bibliography.

On non-Markovian members, in the discussion of the complete-positivity conditions in Sec. III: “In general, all that is required is that the total map from some initial state at $t = 0$ to some final time is completely positive ... In this way, one can consider non-Markovian dynamics.”

On cluster separability: it “demands that the coupling constants $W^{\alpha\beta}(z|z'; x, y)$ be delta functions in x, y , and that the coupling to phase space variables z be to the local classical field $z(x)$.”

On autonomy: “In the remainder of the article, we will restrict our attention to the autonomous case where the coupling constants have no explicit time dependence.”

The companion analysis states that in the non-Markovian case the couplings “are less constrained” [35]. The generator-level positivity conditions of Appendix A, Theorem 1 of Ref. [18], labelled (133)–(135) in arXiv v3, characterize Markovian generators (PRX numbering not checked).

6.2 The chain

Take $C \geq 2$ sites. Each site carries one classical cell with values in $\{\emptyset, 1, \dots, 6\}$. A sits at site 1 and B at site C . There is no sink: erasure writes the blank back. The Kraus operators K_m and quasi-states Y_m are those of Sec. 4. There are five unit windows, with $R_2 = 14$, $R = 12$, $\eta = 6$, $\kappa = 6$:

- W_1 : flush every cell at rate R_2 ;

- W_2 : if cell 1 is blank, write m with $\mathcal{K}_m$ on A at rate R ;
- W_3 : transport, a directed nearest-neighbour hop of a record from cell i into a blank cell $i + 1$ (towards site C) at rate η , with trivial quantum content;
- W_{3b} : flush every cell except site C at rate R_2 ;
- W_4 : if cell C holds m , reset it to blank with $\tilde{\mathcal{G}}_m$ on B at rate κ .

Every quantum jump acts at a single site, so the Lindblad sandwiches are diagonal in x, y . Every rate is keyed to the local cell. The classical transport kernel couples adjacent cells. A strict delta reading applied to the classical sector would forbid classical propagation altogether, so we adopt the nearest-neighbour kernel as a lattice discretization of local classical transport. This is a declared mapping assumption. The generators are one-sided and preserve the block-diagonal register algebra. The undressed family is not positive on all hybrid inputs, so it does not establish single-time classicality as defined in Sec. 2; the blank-register numerical trajectories have a narrower preparation scope. For example, prepare the register in $(\emptyset, 5)$, A in $|0\rangle$ and B in the eigenvector of Y_5 orthogonal to its minimum-eigenvalue eigenvector y ; in output register $(1, \emptyset)$ the product test with $|0\rangle \otimes y$ has weight $-p_1(1 - e^{-R})e^{-2R_2}(1 - e^{-\kappa})\Lambda_\star \approx -1.708 \times 10^{-12}$ (a derivation by the independent referee, Sec. 8). The chain is not a physical model.

6.3 Properties

Proposition 2 (transplant, as built; numerical). *The following hold for the chain as built.*

- From the blank register and $|+\rangle|+\rangle$ the endpoint negativity is 0.386 at $C = 2$ and 0.377 at $C = 3$. For $C = 3$ this was computed on the full 343-state register, and a 12-restart search over product inputs gave a best-found value of 0.378 (not an upper bound).*
- From the blank register and a product input, with $\eta = 0$ (no transport) the endpoint is exactly separable. Without transport, B 's cell never receives a record.*
- The family is not CP-divisible. The W_4 generator's $(\emptyset \leftarrow m)$ block at site C is $\kappa \tilde{\mathcal{G}}_m$, with minimum Choi eigenvalue $-2\kappa\Lambda_\star = -(\sqrt{6445} - 6) \approx -74.281$ at $\kappa = 6$. It therefore violates the Markovian positivity conditions of Ref. [18] while conserving probability. The W_4 propagator alone has minimum block Choi eigenvalue $-2(1 - e^{-\kappa})\Lambda_\star = -12.349$.*
- Replacing the conditionals by CP maps moves W_4 into the Markovian cone and gives $N = 0$ exactly, as Theorem 1 requires.*

Items (a)–(d) are reproduced numerical facts (Sec. 10); (b) and (d) also follow from the structural arguments given. When the transport budget is scaled with distance (ηd_3 with $d_3 = (C-1)/3$), the $|+\rangle|+\rangle$ negativity rises from 0.305 at $C = 2$ to 0.379 at $C = 8$ (for $C \geq 5$ through a sector-reduction argument in the project record; no input search was run there). The Choi spectrum on the full product space was computed only at $C = 2, 3$, where it shows that the chain is not CP, with the negative Choi eigenvalues of Conjecture 1 and no others below the floating floor; for $C \geq 4$ that statement rests on an architecture argument in the project record. We also record the input dependence of the arrival threshold. At $C = 3$, every separable input stays separable when the record's arrival probability is $q \leq 0.033$. This certificate was recomputed on 22 September; it rests on a polytope-support lemma in the project record, and one floor-rounding step reads a stored intermediate file (Sec. 10). The full model is entangled at $q = 0.0335$, and a maximally entangled input is already NPT at $q = 0.0017$. No input-independent threshold is claimed.

6.4 The structural deficit of the undressed chain

The chain as built is *not* exactly CP at finite flush rate. The factorized register creates a corner that the single register of Sec. 4 does not have. Suppose cell C already holds a stale record m that survives the flush (e^{-R_2}), and A 's fresh record m'' survives the flush of site 1 (e^{-R_2}). Then the erase fires with $\tilde{\mathcal{G}}_m$ in the block $((m'', \emptyset) \leftarrow (\emptyset, m))$. The only positive contribution in that block comes from the path in which the stale record was flushed, and on that path B is untouched. Its Choi matrix is proportional to $|\Omega\rangle\langle\Omega|$ and has rank one, so it cannot cover negative directions of $Y_m \otimes \mathbb{K}$ orthogonal to $|\Omega\rangle$.

We state the exact deficit as a conjecture: its proof is not included here, and the evidence is numerical.

Conjecture 1 (exact deficit). *For the five unit windows and precisely the generators specified above, at $C = 2$, every positive rate point and every m'', m , the block $((m'', \emptyset) \leftarrow (\emptyset, m))$ of Φ_5 has minimum Choi eigenvalue exactly*

$$(1 - e^{-R}) (p_{m''} + q_{m''}) e^{-2R_2} (1 - e^{-\kappa}) \lambda_{\min}(Y_m).$$

Hence the composite Choi deficit is at least $c(2)e^{-2R_2}$ with $c(2) = (1 - e^{-R})(1 - e^{-\kappa})(\sqrt{6445} - 6)/24 = 3.087341$. For this schedule, exact complete positivity at finite rates forces every $Y_m \geq 0$, and then $N = 0$ by Theorem 1.

A proof sketch, an enumeration of the two path classes that reach this block from the window keying, is in the project record. Two independent agent referees of another provider derived the $C = 2$ formula for this five-window schedule (Sec. 8). The formula is restricted to the specified schedule; the former ingredient-only class-wide formulation is false. Adding a finite, purely classical flush of cell C from record to blank at rate $\xi = 1$ during W_{3b} changes no quantum jump and keeps five unit windows, but multiplies the negative coefficient of the exposed block $((j, \emptyset) \leftarrow (\emptyset, m))$ by $e^{-\xi}$ while its shield stays a nonnegative multiple of the identity channel; for $j = 1, m = 5$ the minimum Choi eigenvalue changes from $-2.134710935350370 \times 10^{-12}$ to $-7.853162659592610 \times 10^{-13}$ (independent referee's computation, Sec. 8). At the default rates the measured worst eigenvalue is $-c(2)e^{-2R_2}(1 + O(e^{-R_2}))$, with fitted log-slope -2.001 for $C = 2$ over $R_2 = 6-17$ (asymptotic slope -2). At $R_2 = 14$ it equals $-2.135 \times 10^{-12} = -c(2)e^{-28}$, so the tiny negative eigenvalue seen at default rates is the structural deficit, not solver noise. For $C = 3$ the prefactor is $c(3) = c(2)(2 - (2 + x)e^{-x})$ with $x = \eta d_3$, which is 6.113460 at the defaults, where the transport window has the default duration $d_3 = 1$ ($x = 6$); the distance-scaled schedule above, with $d_3 = 2/3$ at $C = 3$, is a different schedule. This value is supported by keying, exact residuals at three rate points and two sweeps, not by a full enumeration. The $c(2)$ prefactor agrees with multiprecision fits to 89 digits. The limit $R_2 \rightarrow \infty$ is exactly CP with $N = 0.386$. That limit is an instantaneous flush, a time-dependent unbounded generator outside the hypotheses of Theorem 1. Outside this schedule the unqualified impossibility is false. A member with reduction-map conditionals is exactly CP at finite rates, but it has $N = 0$. Whether exact complete positivity and $N > 0$ can coexist in the larger undressed class is open.

6.5 A dressed member with numerical evidence for CP

Add one window W_{1a} before the flush, of unit length and rate $\lambda = 6$. When cell C holds a record, B is depolarized by $\mathcal{R}(X) = \text{tr}(X)\mathbb{K}/2$. This is a legal Lindblad dissipator: its jumps are single-site Pauli operators, it is keyed to the local cell, it is one-sided, and it moves no register state. Because $\mathcal{G}_m \circ \mathcal{R} = \mathcal{G}_m$, the dressing does not soften the erasure. Instead it replaces the rank-one shield of the exposed block by a full-rank one of weight proportional to $(1 - e^{-\lambda})/2$.

Proposition 3 (dressed member; numerical at the default rates). *Let $\rho_\star = (1 - e^{-R_2})e^{-\eta}e^{R_2}/(1 - e^{-\kappa})$. At the default rates the dressed two-site member is TP. Floating-point endpoint and time-grid checks are consistent with CP, and an interval enclosure gives a positive exposed-block margin; CP of the full family at every time remains conjectural. We conjecture that it is hybrid CPTP at every $t \in [0, 6]$ whenever*

$$(S1) \quad \rho_\star \frac{1-e^{-\lambda}}{2} \geq \Lambda_\star, \quad (S2) \quad \frac{\nu}{2}(1 - e^{-R_2})(1 - e^{-R}) \geq 2\Lambda_\star e^{-R_2},$$

At the default rates the margins are numerically factors 240.8 (S1) and 2428.5 (S2). Numerically, inside S2, S1 is sharp: when it fails, Φ_6 is not CP. At $\lambda = 6$ the boundary is $R_2^\star = 8.51629$. The physical pathway from the blank register is unchanged, because cell C is blank during W_{1a} . The negativity is 0.386 (floating-point difference exactly 0), the $\eta = 0$ and CP-erasure controls give $N = 0$, and non-divisibility is unchanged.

Condition S2 is Eq. (5). A four-part closed-form certificate for S1 $\wedge$ S2 is in the project record, together with a zero-code-reuse cross-check and an interval enclosure $+5.1187 \times 10^{-10}$ of the exact exposed-block margin. That scalar enclosure is not an interval proof of the full Choi spectrum. The certificate is not reproduced here and has not been independently reviewed, and a margin on one block bounds neither the unexamined blocks nor common nullspaces at all times; this is why both the default-rate and the rate-region statements are conjectures.

6.6 Consequences

Corollary 2 (no local measurement-feedback unravelling). *With initially separable local ancillary resources and local free evolution, neither the family of Theorem 3 nor the chain members of this section admit an unravelling as continuous measurement with measured operators each acting on a single factor followed by feedback operators each acting on a single factor. Such dynamics preserves product states trajectory by trajectory (Remark 7 of Ref. [36]), and these families entangle.*

Scope relative to Ref. [18]. In Sec. VI of that article the null-result sentence reads “the locally correlated realisations considered here predict a null result” for GIE experiments, and “considered here” refers to the autonomous case. On a reading of “locally correlated” that constrains couplings only, the chain’s formal ingredients resemble it: its quantum jumps satisfy the delta condition quoted above, and its classical transport is the declared nearest-neighbour kernel rather than a delta. A formal resemblance does not establish membership in the named physical theory. If “locally correlated” also requires that the classical degrees of freedom can be monitored without disturbing the quantum system, the chain lies outside it (Sec. 5), and this remark lapses. It lies outside “considered here”: it is time dependent and not CP-divisible. Its total map is not CP, with the stated negative Choi eigenvalues, while the dressed member is numerically consistent with CP; exact all-times legality is unestablished, and total-map CP is the article’s legality condition for non-Markovian members. The chain motivates testing a time-dependent, non-divisible extension of the formal generator ingredients under our declared nearest-neighbour mapping. Because exact dressed CP and membership in the named theory are not established, it does not establish an extension or failure of that article’s physical null-result prediction. For finite registers and bounded generators, autonomous mediated members are CP-divisible semigroups and cannot entangle (Theorem 1); the continuous-register case is not covered here. Oppenheim and Sajjad note that at shorter distance scales “the dynamics can be highly non-Markovian” [37], and Fabiano *et al.* exclude this sector by assuming time locality, remarking that CQ gravity is non-local in time “in principle” [16].

7 Discussion: what GIE experiments can and cannot infer

Take the three structural assumptions: a mediator that is a finite classical register at each single time, a generator that is one-sided at every instant, and complete positivity at every time. Theorems 1, 2 and 3 separate what these assumptions buy with and without divisibility. With CP-divisibility, observed entanglement refutes the model class, also for probes with local ancillas. For probes without ancillas, positivity of every intermediate map, and even its product-tested form, is already enough (Theorem 2); what an entangling member must violate is therefore positivity of intermediate maps, not only their complete positivity. Without divisibility, the three assumptions do not forbid entanglement (Theorem 3). Divisibility is sufficient, not necessary (Remark 3). Within the finite-register snapshot-classical model class studied here, excluding these formal examples needs an additional dynamical or operational premise, about memory or, relatedly, about multi-time classicality of the register (Sec. 5); standard classical-mediator no-go theorems already include such premises. For bare product probes, Eq. (1) states a premise of this kind that suffices: nonnegative joint weights for reading the register at one time, resetting the two systems to product states, and measuring them in product bases at a later time, under the explicitly assumed post-reading/reset continuation law. The time-local a.e. ODE representation used here is not a memoryless operational process.

Corollary 3 (memory). *Let (Φ_t) be entangling on $\mathfrak{A}_S$. There is no finite memory register S' , probability vector σ on S' independent of the AB input, and CP-divisible mediated dynamics (Ψ_t) on $\mathfrak{A}_{S \times S'}$ such that, for all t and ϱ , $\Phi_t(\varrho)$ is the marginal on S of $\Psi_t(\varrho \otimes \sigma)$, that is $[\Phi_t(\varrho)]_s = \sum_{s'} [\Psi_t(\varrho \otimes \sigma)]_{(s,s')}$.*

Proof. By Theorem 1, Ψ_t preserves blockwise separability on $S \times S'$. The input $\varrho \otimes \sigma$ is blockwise separable when ϱ is, and summing blocks over s' preserves separability. $\square$

Corollary 3 is an elementary consequence of Theorem 1. It restates, for finite registers, the containment recorded in the project as “entangling snapshot-classical mediated families have non-classical memory in the dilation sense”. A memory-based witness of nonclassical gravitational dynamics is already in print [38]. The corollary suggests what an experiment can certify. On the three structural assumptions, GIE certifies either a nonclassical mediator or a mediator whose memory admits no CP-divisible mediated description on a larger finite classical register. The Tomizuka–Takeda reading [17] realizes the second option with a quantum environment. Tomizuka–Takeda motivates an analogous hidden-quantum interpretation here; we do not construct a compatible quantum-environment dilation of these finite families. Multitime notions of classical memory are developed in Refs. [29, 30, 31]; we have not checked which of their specific theorems apply to this setting.

Decomposability. Ganardi *et al.* define classical mediation for open systems as decomposability $\lambda = \lambda_{BM} \circ \lambda_{AM}$ and argue that quantum-classical states at all times together with decomposability preclude entanglement gain [5]. The family of Theorem 3 has a register that is classical at each single time. If that is read as their state condition, the trajectory condition alone does not establish non-decomposability under their unrestricted quantum-channel definition; our proved exclusion applies to classical-algebra-preserving factors. Their Eq. (19) takes a supremum over the states allowed in the theory, while the decomposability of their Eq. (2) is over unrestricted quantum channels; Corollary 1 excludes decompositions within the classical-register algebra, not decompositions with quantum mediator states at hidden intermediate stages. There is no mathematical conflict. The

difference is between the factor state space (classical-register algebra, as in Corollary 1, or unrestricted quantum channels with possibly quantum hidden states, as in their Eq. (2)) and the set of states whose classicality is imposed (the snapshots of the given family, or all theory-allowed states, as in their Eq. (19)). Every rescue of the no-go that excludes our construction has to state a dynamical axiom, such as divisibility, round mediation, classical memory or multi-time readability of the register. The standard operational notions of a classical mediator already contain such an axiom, so no published theorem is contradicted; stating it explicitly is what this paper argues for. The pinpoints were read in the arXiv text of Ref. [5] by two agent referees; no human has checked this reading. A finite probe by the second referee (an unrestricted AM CNOT followed by a BM SWAP has a classical endpoint mediator and AB negativity 0.5, with a quantum hidden stage) shows why snapshot classicality does not settle unrestricted decomposability; it is not a decomposition of our family.

Named models. We make no statement about any named model. Markovian semigroup models such as Refs. [39, 40] are CP-divisible by construction. Whether their unconditional generators are one-sided in the sense of Definition 2 is a question of presentation that we do not settle. Trillo and Navascués show that the Diósi–Penrose model entangles through a two-body dissipator [10], an escape through one-sidedness rather than divisibility. Angeli and Carlesso’s entangling Markovian models are likewise the complement of ours: they give up one-sidedness and keep divisibility [11].

8 Verification status

Table 1 gives, for every theorem, lemma, proposition, corollary, conjecture and example, the kind of argument and the independent check that applies. “Independent” means a model or program that did not produce the result. “Astra” is GPT-6 Astra (OpenAI); “Sol” is GPT-6 Sol (OpenAI) and “Sol-6.1” is GPT-6.1 Sol (OpenAI, through Codex); the results themselves were produced by Anthropic models (Sec. 11). The full-paper review by Astra (23 September 2026) found the finite-register core sound and returned a MAJOR verdict on framing; its errors E1–E4 and its overview-wording point G3 were corrected. The resulting revision was independently reviewed by Sol-6.1 on 30 September 2026, which returned HOLD pending a listed set of corrections; those corrections are applied in this version. Those corrections were independently delta-reviewed by codex-pro (GPT-6.1 Sol) on 30 September 2026, recommending SEND for an irsi.ai preprint; this is an agent recommendation, not peer review or the author’s release decision. No item has a human check.

9 Limitations

1. *Toy models.* The families are finite-dimensional and tuned: a chosen Kraus family, chosen rates and unit windows. There is no continuum limit, no spacetime and no dynamics derived from any theory of gravity.
2. *Time dependence is essential.* Autonomous versions with bounded generators on finite registers cannot entangle while remaining CP (Theorem 1). A clocked autonomous variant built in the project record fails as the theorem requires: with the non-CP conditionals kept, CP breaks at order unity on the sampled time grid (block eigenvalues -6.06 to -10.6 , with $N = 0.211$ on the illegal trajectory); with the conditionals dominated into the cone, $N = 0$.

Item	Argument	Independent check	Lean
Blockwise hybrid CP $\Leftrightarrow$ CP on $\mathfrak{A}_S$ (Sec. 2)	paper	paper proof; independently assessed by Sol-6.1 (30 Sep); no formal proof	no
Thm. 1	paper	Astra full-paper review: sound; Sol-6.1 review (30 Sep): correct	no
Lemmas 1–4	paper	Astra full-paper review: correct; Lemma 2 also 300 random numerical instances (drafting agent, not independent); Sol-6.1: correct, with the non-TP bridge of Lemma 3 now written out	no
Thm. 2; Lemmas 5–7	paper	Astra recompute (30 Sep): steps (a)–(c), (e) pass, (d) failed on a trace hypothesis (gap D1); D1 repaired; Astra re-check of the repair: pass; Sol-6.1 (30 Sep): integration matches post-D1, correct. Random-model numerics are sanity checks only	no; qubit cores of Lemma 6 reported in a separate project module, not re-checked here
Ex. 1	paper, closed forms	Astra recompute: pass (symbolic spectra, sharpness of $\kappa \geq 4g$); Sol-6.1 independent algebra agrees	no
Rem. 4(i)	computation	Astra recompute: -1.6044 at $t - s = 0.05$ (the product-test value, half the Choi minimum)	no
Rem. 4(ii)–(iv)	paper	Sol-6.1: ancilla limitation in (ii) correct; P/CP equivalence requires TP; (iv) holds a.e.; transpose strictness in (iii) correct for two qubits	no
Rem. 3	paper	Astra full-paper review: correct (CP part); product-positive extension assessed correct by Sol-6.1	no
Thm. 3(a)–(d)	Lean certificates plus paper bridges	Lean package accepted in narrowed form by an agent checker (Claude, 23 Sep); Astra review: correct; zero-code-reuse reimplementaion (Claude); decimals rechecked by Sol and Astra; Sol-6.1 finite core probes agree. Lean covers the listed certificates, not the full theorem text or the post-intervention process	listed declarations (Sec. 4.1)
Prop. 1	Lean plus paper	exact rational replay of the independent certificate (Astra); Astra and Sol-6.1 reviews: correct	Choi PSD: lemmaE_general, lemmaE; TP: Phi_TP; Choi/direct-sum reading is a paper bridge
Cor. 1	paper	Astra and Sol-6.1: correct, for decompositions within the classical-register algebra; unrestricted decomposability undetermined	no
Two-time quasi-probability -0.3087 (Sec. 5)	closed form, numerical	Astra recomputed; Sol-6.1 agrees; valid only under the stated post-reading continuation law	no
Prop. 2	numerical	(a), (c) values reproduced by Sol and Astra; Astra: (b), (d) follow; (a) is a best-found value, not a proved supremum; Sol-6.1: undressed chain not positive on a bare stale-register input (Sec. 6)	no
Conj. 1	conjecture	Astra derived the $C = 2$ block formula for this five-window schedule; Sol-6.1: fixed five-window formula correct, former ingredient-only class-wide claim refuted by an extra classical flush	no
Prop. 3	numerical; default all-times CP and general rate region conjectural	Astra floating-point check of all 2401 endpoint blocks (min -5.6×10^{-16}), not rigorous; one-block interval enclosure is not full-family certification	no
Cor. 2	paper	Astra and Sol-6.1: correct with separable local initial resources and local free evolution, now in the statement	no
Cor. 3	paper	Astra and Sol-6.1: correct in the finite, bounded mediated classical-dilation class; not a general process-memory theorem	no

Table 1: Verification status. Records: `review/REVIEW-astra-full-wave2/REVIEW.md`, `review/NUMCHECK-sol-2026-09-23.md`, `paper/lemmas/b05/recompute-codex/AUDIT.md`, `paper/lemmas/b05/d1-recheck-codex/AUDIT.md`, `lean/LEAN-REPORT.md`, `paper/REFeree.md` (Sol-6.1), all under `papers/qgmine`, and `lanes/done/ACCEPTANCE-lean-2026-09-23.md`, in the project repository.

3. *Non-CP conditionals.* One can object that a register whose update applies a non-positive map to B is not “classical dynamics”. The objection has operational content: under the assumed post-reading continuation law, an intermediate reading of the register, joined with a later measurement on B , has negative joint quasi-weights (Sec. 5). We do not claim that any published no-go theorem needs a further exclusion: on our reading, those cited assume rounds of CP channels, Markovian dynamics or a readable register, each of which excludes this construction.
4. *Quasi-probability kinship* (Sec. 5) is unanalysed.
5. *Scope of the transplant.* The nearest-neighbour classical kernel is a declared mapping assumption. The undressed chain is not CP, with negative Choi eigenvalues of order ce^{-2R_2} , and is not positive on all hybrid inputs. The dressed evidence is numerical and covers $C = 2$, this register family and the default rates only; all-times CP is not certified even there, the rate region $S1 \wedge S2$ is conjectural, and no dressed $C \geq 3$ member has been built.
6. *Scope of Theorems 1 and 2.* They cover finite registers and locally bounded generators. Continuous registers, which every classical-gravity master equation carries, and singular generators are not covered. Theorem 2 covers inputs without ancillas only.
7. *Verification is agent-internal.* Every construction, attack, independent reimplementations and rerun was done by AI agents in one harness. Reruns cannot catch a conceptual error that all implementations share, for example the Choi convention, the definition of the single-time classical state space, or the reading of the positivity cone of Ref. [18]. The project record contains a Choi-convention test on textbook maps; it is not a substitute for a human check.
8. *Novelty search.* An agent-run sweep (arXiv API keyword and author queries for January 2025 to 22 September 2026, forward-citation walks from 14 seed papers, and full reads of the closest eight papers) found no construction of this kind. It is not exhaustive. Journal-only venues were not swept, and a Semantic Scholar keyword pass was partly rate-limited.

10 Reproducibility

The pre-B05 draft’s numerical results were recomputed on 22 September 2026 on a second machine from the original scripts, with Python 3.12.3, numpy 2.5.2, scipy 1.18.1, mpmath 1.3.0 and sympy 1.14.0. The recomputation used 43 runs; every headline value matched the project’s canonical record to the quoted digits. Deviations from a pristine rerun were:

- a hard-coded path prefix was rewritten;
- `referee2_indep.py` ran against a small numpy stand-in for qutip, which was not installed;
- two sweep scripts exit at a plotting import after writing all numbers;
- the $q \leq 0.033$ bound reads one stored intermediate file that was not regenerated;
- the stored judge inputs `S0/DS.npy` and the `S.full` cache were read rather than regenerated (the $q \leq 0.033$ certificate itself was regenerated from a fresh `S0.npy`).

The figures for Theorem 2 and Example 1 were separately checked on 30 September 2026. Separately, another agent in the same harness wrote an independent reimplementations of Theorem 3 from the mathematical specification alone, without opening project code. It reproduces

$N(3) = 0.388021039073148$ inside a certified interval of width 2×10^{-15} , verifies Proposition 1 in exact rational arithmetic, and reports 0 failures. The Lean 4 package `papers/qgm/lean` certifies the content listed in Sec. 4.1; an agent checker rebuilt it from a fresh clone on 23 September 2026 (0 sorry, standard axioms only, kernel replay of the seven project modules; the full Mathlib import closure was not freshly replayed) and accepted it in that narrowed form (acceptance receipt `lanes/done/ACCEPTANCE-lean-2026-09-23.md`). Earlier exact-arithmetic and Lean files from August 2026 in the project record are not relied on here. Table 2 lists the scripts; full SHA-256 hashes are in `SHA256SUMS` next to this file.

Claim	Script (papers/qgm/repro/orig/...)	Value	SHA-256 (prefix)
Thm. 3(d), CP grid	2026-08-04-pilot/artifacts/ p2_counterexample2.py	$N = 0.38802$	e7554edb0bf58ff6
Prop. 1	.../p2_alltimes_lemma.py	verified	ff3bdb4bac9cb8fb
Thm. 3(c)	.../referee2.indep.py (qutip stand-in)	-3.2087	c99b8f6504d3793b
Prop. 2(a,b), $C = 2$	2026-08-11-transplant/seed/ seed_twoocell.py	0.386; $N = 0$ (floating floor)	0704e53ac0ad9437
Prop. 2(a), $C = 2, 3$	.../b1-chain/chain.py; .../xcheck/xcheck_model.py	0.386; 0.377	6b28d1384842ab31; 9181b7fa45a7d71a
Product-input search, $C = 3$	.../xcheck/xcheck_nmdefault.py	0.378	211e6ef146d01866
Prop. 2(c)	.../b2-oppform/oppform.py; .../atk-b2/atk_cone.py	-74.281	62c2356ec569e936; b11dbafb6428b936
Prop. 2(d); clock	.../atk-b1/atk_controls.py; .../atk-b1/atk_clock.py	$N = 0$; 0.211	169626de1cdd8126; 17ed8303f271a061
Conj. 1	2026-08-12-epsilon/b-a2/enum_paths.py, deficit_c2.py, deficit_c3.py; b-mp/mp_c0.direct.py	3.0873; 6.1135	see SHA256SUMS
Reduction-map member	.../atk-a2/atk_loophole2.py	$N = 0$	12ea3ca335f17756
Prop. 3	.../b-a1/member.py, cert_constants.py; .../xcheck/xc_member.py	$\times 240.8$, $\times 2428.5$	7fcab41f6679ab82; a570e0d17b7ead0e
$q \leq 0.033$ certificate	.../c-cert/s3_witness.py	$+2.07 \times 10^{-6}$	7bfd06141698947b
Independent reimplementa- tion	papers/qgm/indep/check_certificate.py	0 fail	5be7901990c2ba1c

Table 2: Scripts behind each numerical claim, with rerun values from 22 September 2026. Paths are relative to the project repository; the independent reimplementaion path is given in full.

Availability. The scripts, the independent reimplementaion and the Lean package are held in the project repository under the paths above. They have not yet been archived publicly, and the Lean package does not yet carry a licence file.

11 AI-assistance disclosure

The constructions, proofs and numerical certificates reported here were found by AI agents running in a theory-mining harness that Peter Seelman built and supervises. The August 2026 work used Anthropic’s Claude Fable 5 and Claude Opus 5 through Claude Code. The results were attacked by agent referees, reimplemented by agents with zero code reuse, and re-run by agent judges. On 22 September 2026 an agent reran them on a second machine and ran a novelty sweep, and this manuscript was drafted by Claude Opus 5.5. Theorem 2 and Example 1 were found by Claude Opus 5.5 on 30 September 2026 and checked by GPT-6 Astra (OpenAI, through Codex), which also wrote the full-paper review whose corrections are applied in this revision; GPT-6 Sol rechecked selected numbers. On 30 September 2026 GPT-6.1 Sol (OpenAI, through Codex) refereed the resulting revision and returned HOLD with a list of corrections; this version applies them (the unsupported dressed-chain all-times CP claim is demoted to numerical evidence, the class-wide

deficit conjecture is restricted to the fixed schedule, and interpretive claims are narrowed), and a further GPT-6.1 Sol delta check recommended SEND for an irsi.ai preprint. Agent review is not peer review. Several headline framings from earlier project drafts were retracted by the agents’ own referees, and only the corrected framings appear here. In particular, “CP-divisibility is exactly the dividing line” was retracted in favour of “sufficient and tight, not necessary”.

No human expert has checked any theorem, proof, construction or number in this paper. Agent refereeing is not peer review, and agent reimplementations share one harness and may share conceptual errors. The items most in need of an expert check are Theorems 1 and 2 with their lemmas, Proposition 1, Conjecture 1, Proposition 3, and the readings of Refs. [18, 5, 17]. Section 8 records which check applies to each result.

Author responsibility. Peter Seelman takes responsibility for the contents, citations, code and release decisions. AI assistance and the limits of verification are disclosed in this section; no statement here should be read as a human verification of the proofs.

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A Proof of Theorem 2 and Example 1

This appendix reproduces the proof of Lemma B05 of the project record (file `lemmas/b05.tex`, after the D1 repair) as it was checked by the independent review described in Sec. 8. The only edits are the theorem and example names, the citation of Kossakowski’s criterion, and the omission of a pointer to a separate Lean module. Notation is that of Sec. 3.1.

Call a Hermiticity-preserving map L on $\mathfrak{A}_S$ with blocks $L_{z'z}$ a *product-P generator* if, for all unit $a, x \in H_A, b, y \in H_B$,

$$\langle x \otimes y | L_{z'z}(P_a \otimes P_b) | x \otimes y \rangle \geq 0 \quad (z' \neq z), \quad (6)$$

$$\langle x \otimes y | L_{zz}(P_a \otimes P_b) | x \otimes y \rangle \geq 0 \quad \text{whenever } x \perp a \text{ or } y \perp b. \quad (7)$$

These are linear inequalities in L , so the product-P generators form a closed convex cone $\mathcal{C}$.

Lemma 5 (generator conditions). *If (Φ_t) is product-positive divisible, then $L_u \in \mathcal{C}$ for almost every u .*

Proof. As in Lemma 1: at a Lebesgue point u of L , $h^{-1}(\Phi_{u+h,u} - \text{id}) \rightarrow L_u$. For $z' \neq z$ the identity does not contribute to block (z', z) . For $z' = z$, $\langle x \otimes y | (P_a \otimes P_b) | x \otimes y \rangle = |\langle x | a \rangle|^2 |\langle y | b \rangle|^2 = 0$ when $x \perp a$ or $y \perp b$. Divide Eq. (1) (with $s = u, t = u + h$) by h and let $h \downarrow 0$. L_u is Hermiticity preserving: Φ_s^{-1} is Hermiticity preserving because Φ_s is, hence so is every $\Phi_{u+h,u}$ and their limit. $\square$

Lemma 6 (arrowhead). *Let C_A, C_B be Hermitian on H_A, H_B and a, b unit vectors. If $M = C_A \otimes P_b + P_a \otimes C_B$ satisfies $\langle v | M | v \rangle \geq 0$ for every product vector v , then $M \geq 0$.*

Proof. M annihilates $a^\perp \otimes b^\perp$ from both sides. Write any vector as $ca \otimes b + u \otimes b + a \otimes w + v_\perp$ with $u \perp a, w \perp b, v_\perp \in a^\perp \otimes b^\perp$; then $\langle v | M | v \rangle$ does not depend on $v_\perp$. If $c \neq 0$, choose $v_\perp = c^{-1}u \otimes w$, which makes $v = (ca + u) \otimes (b + c^{-1}w)$ a product vector. If $c = 0$, the cross terms $\langle u \otimes b | M | a \otimes w \rangle = \langle u | C_A | a \rangle \langle b | w \rangle + \langle u | a \rangle \langle b | C_B | w \rangle$ vanish, and $\langle v | M | v \rangle = \langle u \otimes b | M | u \otimes b \rangle + \langle a \otimes w | M | a \otimes w \rangle$ is a sum of two product-vector values. $\square$

Lemma 7 (P-attributability of mediated product-P generators). *Let $L = L^A + L^B \in \mathcal{C}$ be mediated.*

(i) *$L = \tilde{L}^A + \tilde{L}^B$, with $(\tilde{L}^A)_{z'z} = \tilde{\ell}_{z'z}^A \otimes \text{id}_B$ and $(\tilde{L}^B)_{z'z} = \text{id}_A \otimes \tilde{\ell}_{z'z}^B$, where every jump half $\tilde{\ell}_{z'z}^X$ ($z' \neq z$) is a positive map and every dwell half $\tilde{\ell}_{zz}^X$ is Hermiticity preserving and cross-positive ($\langle x | \tilde{\ell}_{zz}^X(P_a) | x \rangle \geq 0$ for $x \perp a$). No trace condition on L is assumed.*

(ii) *If moreover L is trace annihilating ($\sum_{z'} \text{tr } L_{z'z}(X) = 0$ for all z and X), the halves in (i) can be chosen so that, in addition, $\tilde{L}^A$ and $\tilde{L}^B$ are each trace annihilating.*

Proof. Replace each half by its Hermitian part $\ell \mapsto \frac{1}{2}(\ell + \ell^\#)$, $\ell^\#(X) = \ell(X^\dagger)^\dagger$; since L is Hermiticity preserving and $(\ell \otimes \text{id})^\# = \ell^\# \otimes \text{id}$, the sum is unchanged.

Jumps. Fix $z' \neq z$ and unit a, b . Put $C_A(a) = \ell_{z'z}^A(P_a)$, $C_B(b) = \ell_{z'z}^B(P_b)$. Then $L_{z'z}(P_a \otimes P_b) = C_A(a) \otimes P_b + P_a \otimes C_B(b)$, which by Eq. (6) and Lemma 6 is positive semidefinite. Lemma 2(a) holds verbatim with Ω_X replaced by the rank-one projectors P_a, P_b ($d_X = 1$ in its proof). Hence $\theta_-(a) := \min\{\theta : C_A(a) + \theta P_a \geq 0\}$ and $\theta_+(b) := \max\{\theta : C_B(b) - \theta P_b \geq 0\}$ are finite and $\theta_-(a) \leq \theta_+(b)$ for all a, b . Put $\theta^* = \sup_a \theta_-(a) \leq \inf_b \theta_+(b)$. For every a , $(\ell_{z'z}^A + \theta^* \text{id})(P_a) = C_A(a) + \theta_-(a)P_a + (\theta^* - \theta_-(a))P_a \geq 0$, so $\ell_{z'z}^A + \theta^* \text{id}$ is positive (pure states span the positive cone); likewise $\ell_{z'z}^B - \theta^* \text{id}$. Replace the jump halves by these.

Dwells. For $x \perp a$, Eq. (7) with $y = b$ gives $\langle x \otimes b | L_{zz}(P_a \otimes P_b) | x \otimes b \rangle = \langle x | \ell_{zz}^A(P_a) | x \rangle + |\langle x | a \rangle|^2 \langle b | \ell_{zz}^B(P_b) | b \rangle = \langle x | \ell_{zz}^A(P_a) | x \rangle \geq 0$; symmetrically for B . The dwell halves are left unchanged. This proves (i); no step used a trace condition.

Trace (ii). With $f_X(\cdot) = \sum_{z'} \text{tr } \ell_{z'z}^X(\cdot)$ (after the jump shifts, which cancel in the sum), trace annihilation on $\alpha \otimes \beta$ reads $f_A(\alpha) \text{tr } \beta + \text{tr } \alpha f_B(\beta) = 0$, so $f_A = c_z \text{tr}$ and $f_B = -c_z \text{tr}$, with c_z real because the halves are Hermiticity preserving. Shift $\ell_{zz}^A \rightarrow \ell_{zz}^A - c_z \text{id}$, $\ell_{zz}^B \rightarrow \ell_{zz}^B + c_z \text{id}$; the sum is unchanged, and the shift does not change $\langle x | \ell(P_a) | x \rangle$ for $x \perp a$, so the halves keep the properties in (i). $\square$

Without trace annihilation (ii) can fail while (i) holds: for one register value, $L = \text{id}_A \otimes \text{id}_B$ (halves $\ell^A = \text{id}$, $\ell^B = 0$, $\Phi_t = e^t \text{id}$) lies in $\mathcal{C}$, and $\text{tr} L(P_a \otimes P_b) = 1$, so no decomposition into trace-annihilating halves exists; (i) holds with the given halves, since $\langle x | P_a | x \rangle = 0$ for $x \perp a$.

Proof of Theorem 2. Partition $[0, t]$ uniformly into $I_k = [(k-1)t/n, kt/n]$, $k = 1, \dots, n$, and let $L^{(n)}$ be the piecewise-constant generator equal on I_k to the average $\bar{L}_k = |I_k|^{-1} \int_{I_k} L_u du$. Since $L_u \in \mathcal{C}$ for a.e. u (Lemma 5) and $\mathcal{C}$ is a closed convex cone cut out by continuous linear functionals, $\bar{L}_k \in \mathcal{C}$. It is mediated, with halves the averages of the bounded measurable L_u^A, L_u^B . So part (i) of Lemma 7 applies to $\bar{L}_k$ directly; no trace condition is used here or below, and trace preservation is not a hypothesis of Theorem 2. Unlike the proof of Theorem 1, no measurable choice of a gauge in u is needed. Let $M = \sup_{u \leq t} \|L_u\|$; then $\|L_u^{(n)}\| \leq M$. Interval averaging $A_n : L \mapsto L^{(n)}$ is an L^1 contraction and $A_n g \rightarrow g$ uniformly for continuous g , so approximating L by continuous functions gives $\|L^{(n)} - L\|_{L^1[0,t]} \rightarrow 0$ as the mesh $t/n \rightarrow 0$, and Grönwall's inequality gives $\|\Phi_t^{(n)} - \Phi_t\| \leq e^{2Mt} \|L^{(n)} - L\|_{L^1[0,t]} \rightarrow 0$.

For one factor, with $\tilde{L}^A, \tilde{L}^B$ the halves of $\bar{L}_k$ from Lemma 7(i) and $\delta = t/n$, $e^{\delta \tilde{L}_k} = \lim_{m \rightarrow \infty} (e^{\delta \tilde{L}^A/m} e^{\delta \tilde{L}^B/m})^m$ (Lie product formula, for fixed n). Let $\mathfrak{L}^A$ be the map on $\bigoplus_z B(H_A)$ with blocks $\tilde{\ell}_{z'z}^A$. Then $e^{\tau \tilde{L}^A}$ has blocks $(e^{\tau \mathfrak{L}^A})_{z'z} \otimes \text{id}_B$. Split $\mathfrak{L}^A = D + R$ into block-diagonal and off-diagonal parts. Each dwell half is cross-positive, so $e^{\tau \tilde{\ell}_{zz}^A}$ is positive (Kossakowski's criterion [28]; the cone-invariance theorem for the positive semidefinite cone); R is positive; every term of the Dyson series of $e^{\tau(D+R)}$ is positive, so every block $(e^{\tau \mathfrak{L}^A})_{z'z}$ is a positive map on $B(H_A)$ (none of these steps uses trace preservation). The proof of Lemma 4 uses only positivity of $\psi_{s's}$, so $e^{\tau \tilde{L}^A}(\mathcal{S}_{\text{sep}}) \subseteq \mathcal{S}_{\text{sep}}$; the same holds for B . Products of such maps preserve $\mathcal{S}_{\text{sep}}$, $\mathcal{S}_{\text{sep}}$ is closed, hence $e^{\delta \tilde{L}_k}, \Phi_t^{(n)}$ and Φ_t preserve $\mathcal{S}_{\text{sep}}$. A product input lies in $\mathcal{S}_{\text{sep}}$, and a sum of separable blocks is separable. $\square$

Proof of Example 1. (a) Write $\ell_g = (1-g)\ell_{(1,1,0)} + g\ell_{(1,1,-1)}$ with $\ell_{(\gamma_1, \gamma_2, \gamma_3)}(X) = \sum_i \gamma_i(\sigma_i X \sigma_i - X)$. $\ell_{(1,1,0)} \otimes \text{id}$ and $(\kappa - 4g)\text{id} \otimes \Delta$ are GKLS, hence cross-positive. It remains to show g times $\ell_{(1,1,-1)} \otimes \text{id} + \text{id} \otimes 4\Delta$ is cross-positive. For unit ψ and $x \perp \psi$, with ρ_A, ρ_B the marginals of ψ , $\sum_i \sigma_i M \sigma_i = 2 \text{tr}(M) \mathbb{K} - M$ gives $\langle x | (\ell_{(1,1,-1)} \otimes \text{id})(P_\psi) | x \rangle = 2\langle x | \mathbb{K} \otimes \rho_B | x \rangle - 2|\langle x | \sigma_3 \otimes \mathbb{K} | \psi \rangle|^2$ and $\langle x | (\text{id} \otimes 4\Delta)(P_\psi) | x \rangle = 2\langle x | \rho_A \otimes \mathbb{K} | x \rangle$. So it suffices that, for every traceless Hermitian unitary W ,

$$|\langle x | W \otimes \mathbb{K} | \psi \rangle|^2 \leq \langle x | \rho_A \otimes \mathbb{K} | x \rangle + \langle x | \mathbb{K} \otimes \rho_B | x \rangle \quad (x \perp \psi). \quad (8)$$

In Schmidt bases $\psi = \sqrt{p}|00\rangle + \sqrt{q}|11\rangle$, $x = \sum \xi_{ij}|ij\rangle$, $W = \begin{pmatrix} w_3 & \bar{c} \\ c & -w_3 \end{pmatrix}$ with $w_3^2 + |c|^2 = 1$.

Orthogonality gives $\sqrt{p}\bar{\xi}_{00} = -\sqrt{q}\bar{\xi}_{11} =: \sigma$. Then $\langle x | W \otimes \mathbb{K} | \psi \rangle = 2\sigma w_3 + \sqrt{p}\bar{\xi}_{10}c + \sqrt{q}\bar{\xi}_{01}\bar{c}$ and the right side of Eq. (8) equals $2p|\xi_{00}|^2 + 2q|\xi_{11}|^2 + (p+q)(|\xi_{01}|^2 + |\xi_{10}|^2) = 4|\sigma|^2 + r^2$, $r^2 = |\xi_{01}|^2 + |\xi_{10}|^2$. By Cauchy-Schwarz, $|\langle x | W \otimes \mathbb{K} | \psi \rangle| \leq 2|\sigma||w_3| + |c|(\sqrt{p}|\xi_{10}| + \sqrt{q}|\xi_{01}|) \leq 2|\sigma||w_3| + |c|r \leq \sqrt{(w_3^2 + |c|^2)(4|\sigma|^2 + r^2)}$. Cross-positivity gives positive semigroups $e^{\tau G}$; $e^{u\Delta} \otimes \text{id}$ is CP; so every $\Phi_{t,s}$ is positive. Equality in Eq. (8) at $\psi = \Phi^+$, $x = (\sigma_3 \otimes \mathbb{K})\Phi^+$ shows $\kappa \geq 4g$ is sharp for this pair.

(b) $e^{\tau G} = e^{\tau \ell_1} \otimes e^{\tau \kappa \Delta}$. $e^{\tau \ell_1}$ is the Pauli map with eigenvalues $(1, 1, e^{-4\tau})$ on $(\sigma_1, \sigma_2, \sigma_3)$; its Choi eigenvalues are $2p_i$ with $p_3 = (e^{-4\tau} - 1)/4 < 0$. The Choi matrix of a tensor product is the tensor product of the Choi matrices (up to a permutation), and $J(e^{\tau \kappa \Delta}) \geq 0$ has top eigenvalue $(1 + 3e^{-\kappa\tau})/2$.

(c) For $t \leq 1$, $\Phi_t = e^{t\Delta} \otimes \text{id}$ is CP. For $t = 1 + \tau$, the A factor is the Pauli map with eigenvalues $(e^{-1}, e^{-1}, e^{-1-4\tau})$, whose smallest Pauli weight is $p_3 = (1 - 2e^{-1} + e^{-1-4\tau})/4 > 0$; the B factor $e^{\tau \kappa \Delta}$ is CP. $\square$